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A Second-Order Projection Dynamical Model for Solving Inverse Mixed Variational Inequalities and Application

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INTRODUCTION

Assume that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous operator and $\mathfrak{D} \subset \mathbb{R}^n$ be a nonempty, closed, and convex set. We consider the standard formulation of the variational inequality problem (VIP), which requires determining $u^* \in \mathfrak{D}$ satisfying the following condition

$$\langle \phi(u^*), u - u^* \rangle \geq 0, \quad \forall u \in \mathfrak{D}. \tag{1}$$

We use the well-established projection theorem, where we conclude that the solution of $VI(\phi, \mathfrak{D})$ is $u^* \in \mathfrak{D}$ defined in (1) $\iff$ it has the following projection equation, for a fixed constant $\mu > 0$:

$$u^* = P_{\mathfrak{D}}[u^* - \mu\phi(u^*)], \tag{2}$$

where $P_{\mathfrak{D}} : \mathbb{R}^n \to \mathfrak{D}$ is a projection operator defined by

$$P_{\mathfrak{D}}(x) = [P_{\mathfrak{D}}(x_1), \ldots, P_{\mathfrak{D}}(x_k)]^T,$$

and

$$P_{\mathfrak{D}}(x_i) = \arg\min \|x - v\|, \quad v \in \mathfrak{D}. \tag{3}$$

In this direction, an important extension of VIP is its inverse formulation. Suppose that $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued continuous mapping. If the inverse function $x = \Psi^{-1}(u) = \phi(u)$ exists, then VIP can be formulated as inverse mixed variational inequality (IMVI) is formulated as follows: find $x^* \in \mathbb{R}^n$ such that $\Psi(x^*) \in \mathfrak{D}$ and

$$\langle x^*, y - \Psi(x^*) \rangle + f(y) - f(\Psi(x^*)) \geq 0, \quad \forall\, y \in \mathfrak{D}, \tag{4}$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping and let $f : \mathfrak{D} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function.

If we assume that Ψ satisfies the Lipschitz continuity condition and the second order model (6) establishes the uniqueness and global existence of the solution. Furthermore, we demonstrate that the trajectories of this dynamical model converge globally to the unique solution of the IMVI problem, when Ψ is strongly monotone. Inverse mixed variational inequality problems play a fundamental role in optimization theory, equilibrium modeling, and engineering applications.

In recent years, several authors [1–3] have considered a common form of first-order dynamical models for solving inverse mixed variational inequality problems, which can be written as

$$\begin{cases} \dot{x}(s) = \sigma\Big[P_{\mathfrak{D}}\Big(\Psi(x(s)) - \mu x(s)\Big) - \Psi(x(s))\Big], \\ x(0) = x_0, \end{cases} \tag{5}$$

where $\dot{x} = \dfrac{dx}{ds}$, $P_{\mathfrak{D}}$ denotes the projection operator defined in (3), and $\sigma > 0$ is a fixed parameter. IMVI problems arise naturally in optimization, equilibrium modeling, network flow analysis, economics, engineering systems, and neural network applications.

METHOD

Motivated by these works, we propose a A Second-Order Projection dynamical model with improved stability and convergence properties based on a projection operator. To find a solution to IMVI problem, we consider the dynamical model given below:

$$\begin{cases} \ddot{x}(s) + \tau(s)\dot{x}(s) + \sigma(s)[\Psi(x) - P_{\mathfrak{D}}(\Psi(x) - \mu x)] = 0, \quad s \geq 0, \\ x(0) = x_0, \quad \dot{x}(0) = v_0, \end{cases} \tag{6}$$

where $\ddot{x} = \frac{d\dot{x}}{ds}$, $\tau : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ and $\sigma : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ are Lebesgue measurable functions. The proposed model possesses strong stability properties and leads to an accelerated projection algorithm with provable linear convergence.

Assumptions

1. [4]The operator $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is characterized as follows:
   (a) It is monotone on $\mathfrak{D}$, if for all $x, v \in \mathfrak{D}$,
   $$\langle \Psi(x) - \Psi(v), x - v \rangle \geq 0.$$
   (b) It is strongly monotone on $\mathfrak{D}$ with modulus $\lambda > 0$, if for all $x, v \in \mathfrak{D}$,
   $$\langle \Psi(x) - \Psi(v), x - v \rangle \geq \lambda\|x - v\|^2.$$
   (c) The operator Ψ is called ζ-Lipschitz continuous on $\mathfrak{D}$ if there exists a $\zeta \geq 0$, such that
   $$\|\Psi(x) - \Psi(v)\| \leq \zeta\|x - v\| \quad \forall x, v \in \mathfrak{D}.$$
2. [5] Suppose that $\mathfrak{D}$ is a subset of $\mathbb{R}^n$, which is closed and convex. Then, we obtain the following results:
   (a) In (3), we defined $P_{\mathfrak{D}}$ be the projection operator. Then
   $$\langle x - P_{\mathfrak{D}}(x), P_{\mathfrak{D}}(x) - v \rangle \geq 0, \quad \forall x \in \mathbb{R}^n,\ v \in \mathfrak{D} \subset \mathbb{R}^n. \tag{7}$$
   (b) $P_{\mathfrak{D}}(\cdot)$ is a nonexpansive operator, if the inequality below is satisfied:
   $$\|P_{\mathfrak{D}}(x) - P_{\mathfrak{D}}(v)\| \leq \|x - v\|, \quad \forall x, v \in \mathbb{R}^n. \tag{8}$$
3. [6] For all $x \in \mathbb{R}^n$, $\quad x^* = P_{\mathfrak{D}}(x)$ defined in (3) iff
   $$\langle x^* - x, y - x^* \rangle + \mu f(y) - \mu f(x^*) \geq 0, \quad \forall y \in \mathfrak{D}.$$

Aims and Contributions

- Develop a second-order projection dynamical model.
- Establish existence and uniqueness of strong solutions.
- Prove global asymptotic stability.
- Derive an accelerated projection algorithm.
- Establish linear convergence.

RESULTS & DISCUSSION

Theorem 1 *Let $\mathfrak{D}(\neq \emptyset)$ be a subset of $\mathbb{R}^n$, which is closed and convex. Also, assume that $f : \mathfrak{D} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous function. Suppose Ψ be ζ-Lipschitz continuous and λ be strongly monotone on $\mathfrak{D}$. Then x^* is a solution of IMVI problem (4) if and only if x^* satisfies*

$$\Psi(x^*) = P_{\mathfrak{D}}\big(\Psi(x^*) - \mu x^*\big). \tag{9}$$

From Theorem (1). It is apparent that x^* acts as a solution of the IMVI problem (4) $\iff$ the dynamical model (6) gives an equilibrium point (i.e., within the dynamical model, the function $x(s) \equiv x^*$ remains constant and constitutes a valid trajectory).

Discrete System with Linear Convergence Rate: By applying a finite-difference scheme to equation (6) with respect to the time variable s, and using a step size $h_k > 0$, a relaxation variable $\sigma_k > 0$, a damping variable $\tau_k > 0$, along with initial points x_0 and u_1, we derive the following iterative scheme:

$$\frac{1}{h_k^2}(x_{k+1} - 2x_k + x_{k-1}) + \tau_k\frac{1}{h_k}(x_k - x_{k-1}) = \sigma_k[P_{\mathfrak{D}}(\Psi(x_k) - \mu x_k) - \Psi(x_k)]. \tag{10}$$

Algorithm 1 Accelerated Projection Algorithm from dynamical Approach

Setting: $\theta_k = 1 - \tau_k h_k$ and $\rho_k = h_k^2\sigma_k$ in (10).
Initialization: Choose $x_0 \in \mathbb{R}^n$.
Iteration: For $k = 1, 2, \ldots$, compute

$$x_{k+1} = x_k + \theta_k(x_k - x_{k-1}) + \rho_k[P_{\mathfrak{D}}(\Psi(x_k) - \mu x_k) - \Psi(x_k)]. \tag{11}$$

This algorithm is a relaxed projection, and the convergence properties of (11) are examined in this section. Using the dynamical model (6), we examine the convergence of the trajectories. The next result is fundamental to the convergence analysis.

Theorem 2 *Let $\mathfrak{D}(\neq \emptyset)$ be a subset of $\mathbb{R}^n$, which is closed and convex. Suppose Ψ be ζ-Lipschitz continuous and λ be a strongly monotone on $\mathfrak{D}$. Let x^* be the unique solution of IMVI problem (4). $\forall\mu > 0$ and $x \in \mathbb{R}^n$, denote $z := P_{\mathfrak{D}}(\Psi(x) - \mu x)$. Then*

$$\langle x - x^*, \Psi(x) - z \rangle \geq \alpha_1\|\Psi(x) - z\|^2, \tag{12}$$

where

$$\alpha_1 = \frac{\alpha}{(2\zeta + \mu)^2}; \quad \alpha := \left(\lambda + \mu\lambda - \frac{\zeta^2}{2} - \frac{\mu^2}{2} - \frac{1}{2}\right) > 0,$$

and

$$\langle z - \Psi(x),\ x - x^* \rangle \leq -\alpha\|x - x^*\|^2. \tag{13}$$

Theorem 3 *Suppose $\mathfrak{D}(\neq \emptyset)$ be a subset of $\mathbb{R}^n$, which is closed and convex. Let Ψ be ζ-Lipschitz continuous and λ be a strongly monotone on $\mathfrak{D}$. Assume that*

$$0 < P < \rho_k C < Q, \quad \text{where } C = (1 + \theta_k^2), \tag{14}$$

or equivalently

$$0 < \frac{Q^2}{P} < \frac{2\rho_k C}{(2\zeta + \mu)^2}. \tag{15}$$

Then, the sequence $\{x_k\}$ produced by the algorithm (11) converges linearly to the unique solution x^* of IMVI problem (4).

Application of Lyapunov's Direct Method for Stability Verification: Sppose $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ is considered a Lyapunov function for the dynamical model (6) if it approximately characterizes the equilibrium point $x = x^*$ and satisfies the following conditions:

(C1) The function $\mathcal{V}$ is positive definite; that is, $\mathcal{V}(x) \geq 0$ for all $x \in \mathbb{R}^n$, and $\mathcal{V}(x) = 0 \iff x = x^*$.

(C2) Along the trajectories of the dynamicalal model (6), the time derivative $\dot{\mathcal{V}}$ is negative definite. In other words, for any solution $x(s)$ of (6), it holds that $\dot{\mathcal{V}}(x(s)) \leq 0$ for all $s \geq 0$, and $\dot{\mathcal{V}}(x(s)) < 0$ whenever $x(s) \neq x^*$.

The following theorem represents a fundamental result in dynamical models theory.

Theorem 4 *[Theorem of Lyapunov's] Let x^* denote an equilibrium point of the dynamical model (6). If the Lyapunov function is associated with x^*, then it follows that x^* serves as a globally asymptotically stable state of the model (6).*

We apply Theorem (4) to establish the solution of the model (6) is stable. Then, using Lyapunov's direct method, we verify that the proposed second-order dynamical Model (6) demonstrates asymptotic stability. For this purpose, we consider a Lyapunov candidate function defined as:

$$\mathcal{V}(x) = \frac{1}{2}\left(\dot{x}^2 + x^2\right).$$

CONCLUSION

- We propose a second-order dynamical model for solving inverse mixed-variational inequality problems in Hilbert spaces.
- We establish the existence and uniqueness of strong global solutions under strong monotonicity and Lipschitz continuity assumptions.
- A Lyapunov function is constructed to prove the global asymptotic stability of the proposed dynamical model (6).
- A relaxed inertial projection algorithm is derived from the continuous model (6) and shown to converge linearly.

FUTURE WORK / REFERENCES

Future Work:

- Fixed-time and predefined-time dynamical models.
- Time-varying inverse variational inequality problems.
- Applications in machine learning and network equilibrium models.
- Extension to higher-order projection dynamical models.
- Development of adaptive and distributed dynamical algorithms.

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