Variational Structures in Hamilton–Jacobi Equations and Scalar Conservation Laws

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Viscosity and entropy solutions emerge from the same variational principle.

1. The Problem

Physical phenomena front propagation, wave breaking, optimal control are governed by **first-order nonlinear PDEs**.

The canonical model is **inviscid Burgers**:

$$u_t + u\,u_x = 0, \qquad u(x,0) = u_0(x).$$

Characteristics $x = x_0 + u_0(x_0)\,t$ cross at

$$T^* = \frac{-1}{\min_x u_0'(x)} < \infty$$

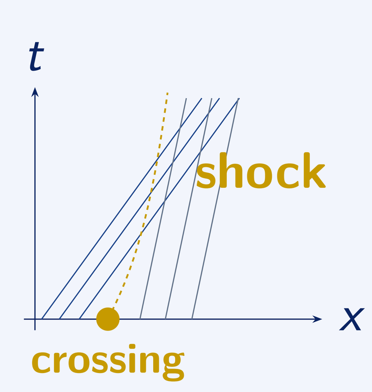
when $u_0' < 0$ somewhere. Classical solutions break down.

Question: *How do global solutions exist beyond T^*, and how are they selected uniquely?*

2. Shock Formation

For $u_t + f(u)_x = 0$, weak solutions develop **shocks**. A discontinuity at speed s satisfies:

$$s = \frac{f(u_R) - f(u_L)}{u_R - u_L} \quad \text{(Rankine–Hugoniot)}$$



Rankine–Hugoniot admits multiple solutions an **admissibility criterion** is required.

3. Entropy Admissibility

The **Kružkov entropy condition** selects the physical solution: for all $k \in \mathbb{R}$,

$$\partial_t|u-k| + \partial_x\big[\operatorname{sgn}(u-k)(f(u)-f(k))\big] \le 0$$

For convex f, this reduces to **Lax's condition**:

$$f'(u_L) \;\ge\; s \;\ge\; f'(u_R)$$

*Characteristics must **enter** the shock.*

4. The Hopf–Lax Formula

Hamilton–Jacobi equation with convex H:

$$u_t + H(Du) = 0, \quad u\big|_{t=0} = g$$

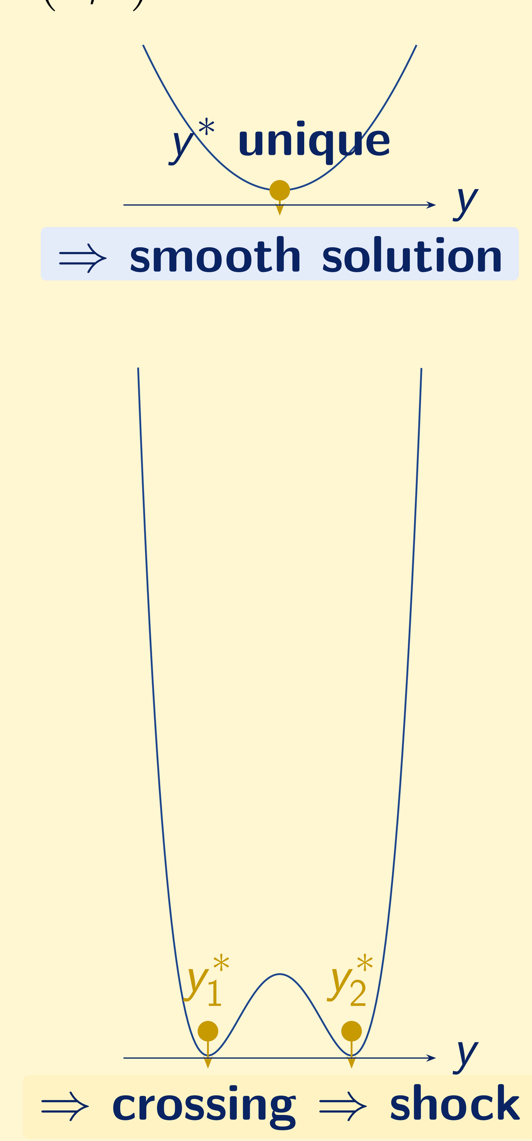
Legendre–Fenchel transform: $L(v) = \sup_p(\langle p, v\rangle - H(p))$.

$$u(x,t) \;=\; \min_{y\in\mathbb{R}^n}\Big\{\, t\,L\Big(\tfrac{x-y}{t}\Big) + g(y) \,\Big\}$$

Theorem: If H is convex and superlinear, this gives the unique viscosity solution with $u(\cdot,t) \in W^{1,\infty}(\mathbb{R}^n)$ for all $t > 0$.

5. Main Structural Observation

The minimiser $y^*(x,t)$ encodes the **solution geometry**:



Viscosity (HJ) and entropy (conservation laws) are two faces of one principle: select the unique admissible minimiser.

6. Lax–Oleinik Connection

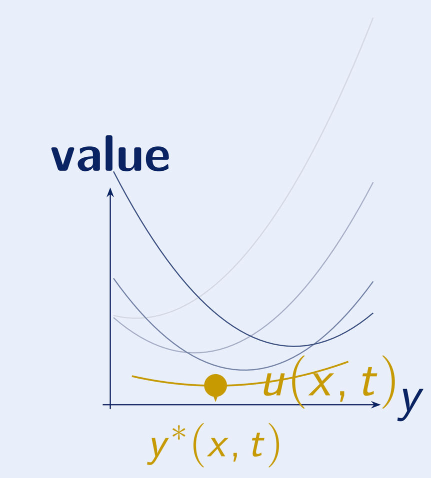
For Burgers ($H(p) = p^2/2$), Hopf–Lax becomes:

$$u(x,t) = \min_{y\in\mathbb{R}}\left\{\frac{(x-y)^2}{2t} + u_0(y)\right\}$$

Both are **inf-convolutions** driven by convex duality.

7. Minimisation Geometry

$u(x,t)$ is the **lower envelope** of cost functions:



- **Unique** y^* $\Rightarrow$ smooth solution
- **Non-unique** y^* $\Rightarrow$ shock formation
- y^* = foot of optimal characteristic

8. Convexity: The Unifying Principle

Convexity of H (or f) simultaneously guarantees:

- **Global existence** minimiser always exists
- **Uniqueness** L^1-contraction property
- **Regularity** Oleinik estimate $u_x \le 1/t$
- **Shock structure** finite Hausdorff measure

When convexity fails, none of these hold automatically.

References

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