IOCMA 2026 Conference

The 2nd International Online Conference on Mathematics and Applications

10–12 June 2026 | Online

MDPI

Four Algebraic Limit Cycles Surrounding the Same Singular Point in a Planar Polynomial Differential System

Fouad Khireddine[1,2], Mohamed Grazem[1,3]

[1]Department of Mathematics, Faculty of Siences, University of Boumerdes, Algeria | [2]Dynamics of Engines and Vibroacoustics Laboratory, University of Boumerdes, Algeria | [3]Laboratory of Applied Mathematics, Faculty of Sciences, Sétif 1 University Ferhat Abbas, Algeria

INTRODUCTION & AIM

A polynomial differential system on the plane has the form

$$\begin{cases}\dot{x}=\frac{dx}{dt}=P(x,y),\\ \dot{y}=\frac{dy}{dt}=Q(x,y),\end{cases}\tag{1}$$

where $P, Q \in \mathbb{R}[x,y]$ are coprime. The degree of (1) is $\max(\deg P, \deg Q)$.

The algebraic curve $U(x,y)=0$ is an *invariant curve* for (1) if there exists a polynomial $K(x,y)$ (the *cofactor*) such that

$$P\,\partial_x U + Q\,\partial_y U = K\,U.$$

A *limit cycle* of (1) is an isolated periodic solution. It is *algebraic* if it lies in the zero set of an invariant curve, otherwise *non-algebraic*.

One of the central open problems in the qualitative theory of planar polynomial systems [5] is the second part of Hilbert's 16th problem: bounding the number $H(n)$ of limit cycles of degree-n systems. Giving explicit expressions for limit cycles is a harder, yet important, companion question; until recently only algebraic cycles were known explicitly [1, 2, 3, 4, 7].

When a system has several algebraic limit cycles, each typically surrounds a *different* singular point. For instance, Bendjeddou and Cheurfa [3] showed that a quartic system can have up to four limit cycles, each encircling a distinct singularity. In this work we exhibit **four algebraic limit cycles surrounding the same singular point** in the system

$$\begin{cases}\dot{x}=-2y(x^2+y^2)+x\,\Phi(x,y),\\ \dot{y}=\quad 2x(x^2+y^2)+y\,\Phi(x,y),\end{cases}\tag{2}$$

where

$$\Phi=x^8+4x^6y^2-16x^6+6x^4y^4-48x^4y^2+77x^4+4x^2y^6-48x^2y^4+154x^2y^2-134x^2+y^8-16y^6+77y^4-134y^2+72.$$

Proposition 1. *The origin is the only singular point of* (2) *and it is an unstable node.*

Proof. $yP-xQ=-2(x^2+y^2)^2\neq 0$ for $(x,y)\neq(0,0)$, so the origin is the unique singular point. The unique eigenvalue of the linearisation is $\lambda=72>0$, hence it is an unstable node. □

Theorem 1. *System* (2) *admits four circles*

$$(\Gamma_1): x^2+y^2=1,\qquad (\Gamma_2): x^2+y^2=2,$$
$$(\Gamma_3): x^2+y^2=4,\qquad (\Gamma_4): x^2+y^2=9,$$

as hyperbolic limit cycles. Γ_1 *and* Γ_3 *are stable;* Γ_2 *and* Γ_4 *are unstable.*

Remark 1. $\Gamma_1\subset\Gamma_2\subset\Gamma_3\subset\Gamma_4$*; all four cycles surround the same singular point (the origin).*

METHOD

Proof. It is easy to check that the system (2) possesses (Γ_k), $k=\overline{1.4}$ as invariant algebraic curve. The associated cofactors are respectively given

$$K_1(x,y)=2(x^2+y^2)\Big(x^6+3x^4y^2-15x^4+3x^2y^4-30x^2y^2+62x^2+y^6-15y^4+62y^2-72\Big),$$

$$K_2(x,y)=2(x^2+y^2)\Big(x^6+3x^4y^2-14x^4+3x^2y^4-28x^2y^2+49x^2+y^6-14y^4+49y^2-36\Big),$$

$$K_3(x,y)=2(x^2+y^2)\Big(x^6+3x^4y^2-12x^4+3x^2y^4-24x^2y^2+29x^2+y^6-12y^4+29y^2-18\Big),$$

$$K_4(x,y)=2(x^2+y^2)\Big(x^6+3x^4y^2-7x^4+3x^2y^4-14x^2y^2+14x^2+y^6-7y^4+14y^2-8\Big).$$

As the all circles (Γ_k) for $k=\overline{1.4}$ do not pass through the origin, thus they are periodic solutions of the system (2) .
Let T denotes be the period of (Γ_k). To show that (Γ_k) is a limit cycle, it is sufficient to check that

$$I\left(\Gamma_k\right)=\int\limits_0^T div\left(x(t),y(t)\right)dt\neq 0$$

According to theorem 3 of [7], we have

$$I\left(\Gamma_k\right)=\int_0^T div\left(x\left(t\right),y\left(t\right)\right)dt=\int_0^T K_k\left(x\left(t\right),y\left(t\right)\right)dt$$

Therefore

$$I(\Gamma_1)=\int_0^T K_1\big(x(t),\,y(t)\big)\,dt=-48$$

which means that (Γ_1) is hyperbolic and stable limit cycle.

$$I(\Gamma_2)=\int_0^T K_2\big(x(t),\,y(t)\big)\,dt=56$$

so (Γ_2) is hyperbolic and unstable limit cycle.

$$I(\Gamma_3)=\int_0^T K_3\big(x(t),\,y(t)\big)\,dt=-240$$

indicating that (Γ_3) is hyperbolic and stable limit cycle.

$$I(\Gamma_4)=\int_0^T K_4\big(x(t),\,y(t)\big)\,dt=5040$$

hence (Γ_4) is hyperbolic and unstable limit cycle. □

Moreover, the circle (Γ_1) lies inside (Γ_2) and the circle (Γ_2) lies inside (Γ_3) and the circle (Γ_3) lies inside (Γ_4) as shown on the Poincaré disc in Figure 1.

RESULTS & DISCUSSION

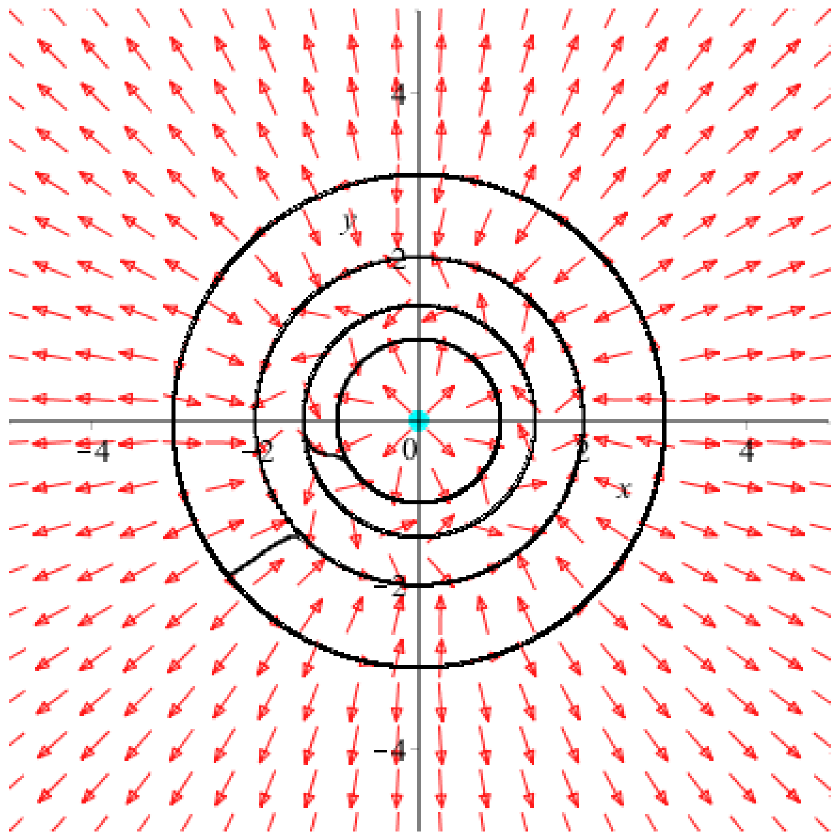

Figure 1 : The phase portrait in the Poincaré disk of the system (2) showing four limit cycles.

CONCLUSION

We have exhibited a planar polynomial differential system possessing four explicit algebraic limit cycles, all surrounding the *same* singular point a configuration not previously reported in the literature. The cycles alternate between stable and unstable. A natural extension is to seek similar coexistence phenomena in classes of Kolmogorov systems.

REFERENCES

References

[1] A. Bendjeddou, A. Berbache, R. Cheurfa, A class of Kolmogorov system with exact algebraic limit cycle, *Int. J. Diff. Equ. Appl.* **14**(3) (2015), 159–165.
[2] A. Bendjeddou, R. Cheurfa, On the exact limit cycle for some class of planar differential systems, *Nonlinear Diff. Equ. Appl.* **14** (2007), 491–498.
[3] A. Bendjeddou, R. Cheurfa, Cubic and quartic planar differential system with exact algebraic limit cycles, *Electron. J. Diff. Equ.* **2011**(15) (2011), 1–12.
[4] S. Benyoucef, A. Bendjeddou, Kolmogorov system with explicit hyperbolic limit cycle, *J. Sib. Fed. Univ. Math. Phys.* **10**(2) (2017), 216–222.
[5] F. Dumortier, J. Llibre, J. Artés, *Qualitative Theory of Planar Differential Systems*, Springer, 2006.
[6] H. Giacomini, J. Llibre, M. Viano, On the nonexistence, existence and uniqueness of limit cycles, *Nonlinearity* **9** (1996), 501–516.
[7] J. Giné, M. Grau, A note on: Relaxation oscillator with exact limit cycles, *J. Math. Anal. Appl.* **224**(1) (2006), 739–745.
[8] J. Llibre, J. Yu, X. Zhang, On the limit cycle of polynomial differential systems with a linear node and homogeneous nonlinearities, *Int. J. Bifur. Chaos* **24**(5) (2014), 1450065.
[9] L. Perko, *Differential Equations and Dynamical Systems*, 3rd ed., Springer, 2001.