

Green's Function for a One-Dimensional Fractional Viscoelastic Wave Equation

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Abstract

We investigate the Green's function associated with a one-dimensional fractional partial differential equation that models the propagation of disturbances in visco-elastic media driven by a Dirac delta source. The starting point of the formulation is the classical one-dimensional viscoelastic wave equation posed with a Dirac delta driving term, which defines the fundamental solution of the corresponding wave operator. Fractional effects are incorporated directly at the level of this forced equation by generalizing the time and space operators: the second-order time derivative is replaced by the Caputo fractional derivative of order $\alpha \in (1, 2]$, while the spatial Laplacian is replaced by the Riesz fractional operator of order $\beta \in (1, 2]$. This construction leads to a fractional wave equation with memory in time and nonlocality in space, formulated explicitly as a Green's function problem.

The Green's function is obtained by applying the Laplace transform in time and the Fourier transform in space to the fractional wave equation, reducing the problem to an algebraic equation in the transform domain. The inverse transforms yield an explicit representation of the fundamental solution in terms of Fox H-function series, which naturally arise from the combined action of the fractional temporal and spatial operators. The resulting expression provides an explicit form of the Green's function for the fractional viscoelastic wave equation with singular forcing.

Motivation

In a previous work, Duffy¹ solved by means of integral transform methods the equation

$$[1 + K(t)*] \frac{\partial^2 g}{\partial t^2} - v^2 \frac{\partial^2 g}{\partial x^2} = [1 + K(t)*] \delta(t - \tau) \delta(x - \varepsilon), \quad (1)$$

for $v = 1$. In Equation (1) * denotes convolution between two functions. The solution $g(x, t; \xi, \tau)$ of Equation (1) corresponds to the *Green's function* of a partial differential equation that models the propagation of disturbances in viscoelastic media². In Equation (1) the function $K(t)$ is given in the next form

$$K(t) = \left(\frac{a}{\sqrt{\pi t}} + a^2 \right) H(t) \quad (2)$$

where $a > 0$ and $H(t)$ stands for the Heaviside step function.

Formulation of the Fractional Model

Inspired by this work, we construct a fractional model of Equation (1) by proposing the following transformation

$$\frac{\partial^2}{\partial t^2} \rightarrow D_t^\alpha, \quad \frac{\partial^2}{\partial x^2} \rightarrow \mathbb{D}_x^\beta, \quad v \rightarrow v_{\alpha, \beta}, \quad 1 \leq \alpha, \beta \leq 2, \quad (3)$$

where D_t^α stands for the α -order Caputo fractional derivative operator, \mathbb{D}_x^β for the β -order Riesz pseudo-differential operator and $[v_{\alpha, \beta}] = [s]^{-\alpha} [m]^\beta$. To cover a wider variety of cases, a generalization for the function $K(t)$ in Equation (2) is also proposed in the next form

$$K_\gamma(t) = \left(\frac{2a^{\frac{\gamma}{2}}}{\Gamma(\frac{\gamma}{4}) t^{1-\frac{\gamma}{4}}} + \frac{a^\gamma}{\Gamma(\frac{\gamma}{2}) t^{1-\frac{\gamma}{2}}} \right) H(t), \quad (4)$$

where $1 \leq \gamma \leq 2$. For $\gamma = 2$, we recover $K(t)$. Introducing the transformations of Equations (3) and (4) into Equation (1) we arrive to the fractional model

$$[1 + K_\gamma(t)*] D_t^\alpha u - v_{\alpha, \beta}^2 \mathbb{D}_x^\beta u = [1 + K_\gamma(t)*] \delta(t - \tau) \delta(x - \varepsilon).$$

In this work we consider initial conditions equal to zero and no boundary conditions.

Problem Solution

The solution is given as the next series

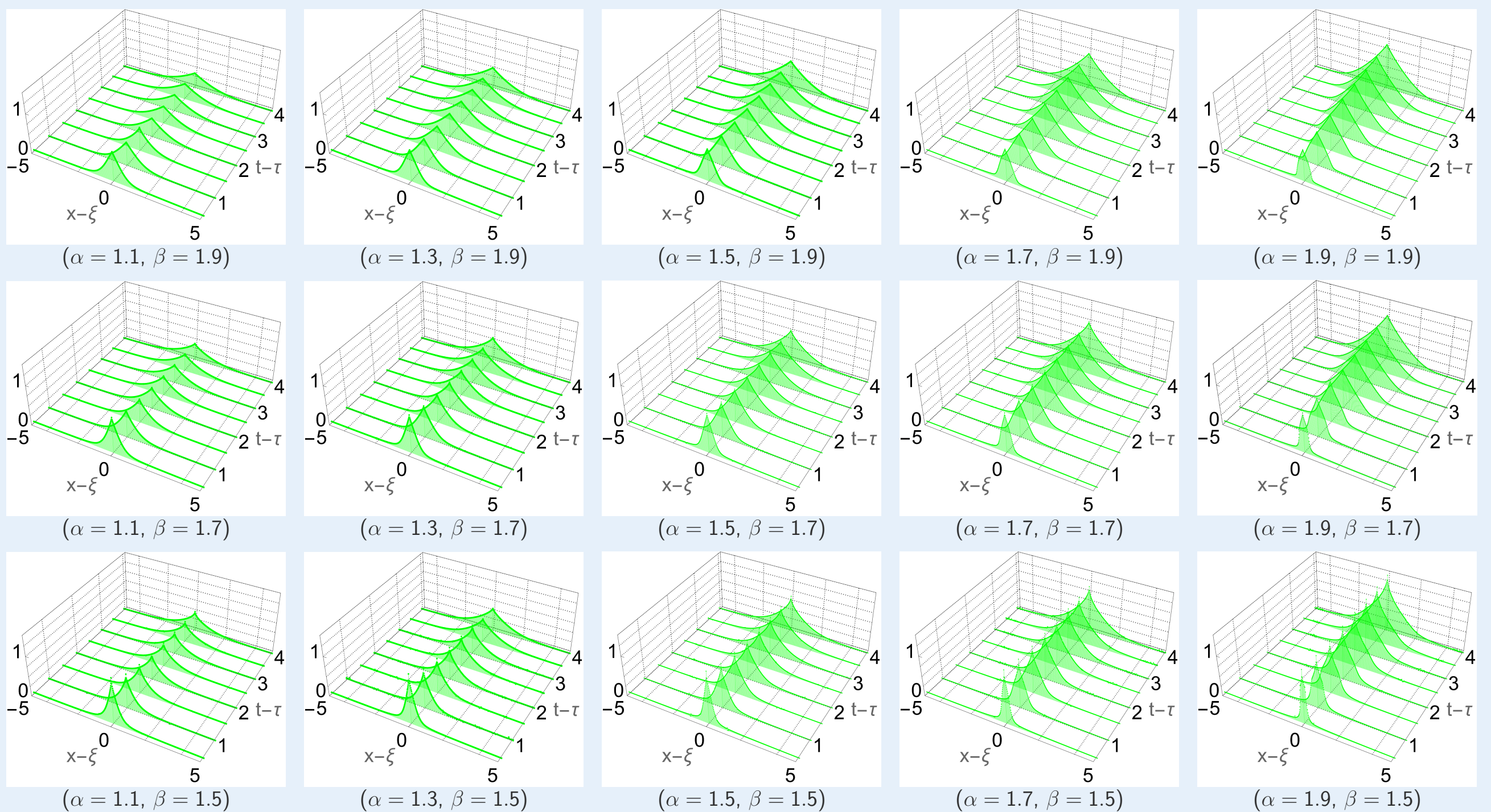
$$g(x, t; \xi, \tau) = \frac{1}{\beta v_{\alpha, \beta}^{\frac{2}{\beta}} (t - \tau)^{1 - \alpha(1 - \frac{1}{\beta})}} \sum_{n=0}^{\infty} \frac{(a^{\frac{\gamma}{2}} (t - \tau)^{\frac{\gamma}{4}})^n}{n!} \Theta_n(x, t; \xi, \tau) \quad (5)$$

where

$$\Theta_n(x, t; \xi, \tau) = H_{4,4}^{2,2} \left[\frac{|x - \xi|}{\beta v_{\alpha, \beta}^{\frac{2}{\beta}} (t - \tau)^{\frac{\alpha}{\beta}}} \middle| \begin{matrix} (1 - \frac{1}{\beta}, \frac{1}{\beta}), (-\frac{2}{\beta}, \frac{2}{\beta}), (\frac{1}{2}, \frac{1}{2}), (\alpha(1 - \frac{1}{\beta}) + \frac{\gamma n}{4}, \frac{\alpha}{\beta}) \\ (0, 1), (1 - \frac{1}{\beta}, \frac{1}{\beta}), (\frac{1}{2}, \frac{1}{2}), (n - \frac{2}{\beta}, \frac{2}{\beta}) \end{matrix} \right]. \quad (6)$$

and $H[-]$ stands for the H-Fox function. The expressions in Equations (5) and (6) describe the Green's function, which also corresponds to the system response to a Dirac delta source.

Graphic Representation of the Solution



Graphical representations of the Green's function for different values of α and β , $\gamma = \alpha$ and $v_{\alpha, \beta} = a = 1$.

Concluding Remarks and Forthcoming work

- ▶ The Green's function is obtained in terms of a Fox H-function series, which depends on the fractional parameters α and β .
- ▶ The graphical representations describe the Green's function, which also corresponds to the system response to a Dirac delta source, showing a triangular profile that progressively broadens as time evolves.
- ▶ Future work will focus on studying the system response to different source terms, using the Green's function obtained here to compute the corresponding solutions through convolution.

References

1. Duffy, D. G. *Green's Functions with Applications*; CRC Press: Boca Raton, United States of America, 2015.
2. Hangya, A., and M Seredynska, 1999: Some effects of the memory kernel singularity on wave propagation and inversion in poroelastic media-I. Forward problems. *Geophys. J. Int.*, **137**, 319-335. Published by Blackwell Science Ltd. Oxford, UK