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A Robust Adaptive Hybrid Nonlinear Conjugate Gradient Method Ensuring Global Convergence

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INTRODUCTION & AIM

Context. Nonlinear conjugate gradient (NCG) methods solve $\min_{x\in\mathbb{R}^n} f(x)$ with $O(n)$ storage and one gradient evaluation per iteration. The update rule is:

$$x_{k+1} = x_k + \alpha_k d_k, \quad d_{k+1} = -g_{k+1} + \beta_k d_k$$

Classical β_k formulas ($y_k = g_{k+1} - g_k$):

$$\beta^{\text{FR}} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \beta^{\text{PRP}} = \frac{g_{k+1}^\top y_k}{\|g_k\|^2}, \quad \beta^{\text{HS}} = \frac{g_{k+1}^\top y_k}{d_k^\top y_k}$$

FR: strong theory, slow convergence. **PRP/HS**: faster but may lack global convergence without safeguards.

Aim. Propose **AHNCG**, a hybrid that:

- Continuously combines FR, PRP, HS via a geometry-driven weight θ_k
- Guarantees *uniform* sufficient descent for all k
- Proves strong convergence $\lim_{k\to\infty} \|g_k\| = 0$

RESULTS & DISCUSSION

Test bench. 96 CUTEst problems, $n \in \{50, \ldots, 10000\}$, strong Wolfe line search ($c_1{=}10^{-4}$, $c_2{=}0.4$), $\varepsilon = 10^{-6}$.

Method	NF	NI	CPU (s)	SR (%)
FR	1247	843	1.82	76.0
PRP$^+$	938	614	1.41	84.4
HS$^+$	902	591	1.37	85.4
DY	1053	702	1.58	81.2
CG_DESCENT	741	481	1.12	90.6
AHNCG	**683**	**441**	**1.03**	**93.7**

NF: gradient evaluations; NI: iterations; SR: success rate.

Key findings:

- AHNCG reduces gradient evaluations by **45%** vs. FR and **8%** vs. CG_DESCENT
- Success rate improved by **+3.1 pp** over the next-best method
- Gains most pronounced on ill-conditioned problems ($\kappa > 10^4$), where θ_k correctly favours the PRP component

METHOD

Adaptive hybrid parameter:

$$\beta_k^{\text{adaptive}} = \theta_k\, \beta_k^{\text{PRP}} + (1-\theta_k)\min\{\beta_k^{\text{FR}},\, \beta_k^{\text{HS}}\}$$

Cosine-similarity weight:

$$\theta_k = \frac{|g_{k+1}^\top g_k|}{\|g_{k+1}\|\,\|g_k\|} \in [0,1]$$

$\theta_k \approx 0$: gradients orthogonal $\Rightarrow$ conservative FR–HS blend.
$\theta_k \approx 1$: gradients aligned $\Rightarrow$ aggressive PRP.

Reinforced safeguard (novel):

$$S_k = \begin{cases} \dfrac{(1-c_2)\,\|g_{k+1}\|^2}{|g_{k+1}^\top d_k|} & g_{k+1}^\top d_k \neq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

$$\beta_k \leftarrow \max\{0,\ \min(\beta_k^{\text{adaptive}},\ \tau,\ S_k)\}$$

Directly bounds $\beta_k\, |g_{k+1}^\top d_k| \le (1-c_2)\,\|g_{k+1}\|^2$, guaranteeing the *exact* descent constant c_2.

CONCLUSION

Theoretical contributions:

1. **Uniform sufficient descent** (Lemma 1): $g_k^\top d_k \le -c_2\,\|g_k\|^2$ for all $k \ge 0$, proved by induction covering the degenerate case $g_{k+1}^\top d_k = 0$.
2. **Bounded directions** (Lemma 2): $\|d_k\| \le \gamma/(1-\tau)$ for all $k \ge 0$.
3. **Strong global convergence** (Theorem 1): $\lim_{k\to\infty} \|g_k\| = 0$ (stronger than the classical $\liminf = 0$ from Zoutendijk alone).

Proof sketch of Theorem 1:

1. Zoutendijk condition: $\sum_k \dfrac{(g_k^\top d_k)^2}{\|d_k\|^2} < \infty$
2. Lemma 1 $\Rightarrow (g_k^\top d_k)^2 \ge c_2^2\,\|g_k\|^4$
3. Lemma 2 $\Rightarrow \sum_k \|g_k\|^4 < \infty$
4. Non-negative series convergence $\Rightarrow \|g_k\|^4 \to 0 \Rightarrow \|g_k\| \to 0$ $\square$

FUTURE WORK / REFERENCES

Future work:

- Quasi-Newton corrections to β_k
- Extension to constrained optimization
- Stochastic variants for large-scale machine learning

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