Existence of Global Positive Solutions to Order-[m] Tridiagonal Reaction-Diffusion Systems via Semigroup Methods

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INTRODUCTION AND AIM

Reaction-diffusion systems with tridiagonal diffusion matrices arise naturally in various applications, including population dynamics, chemical reactors, and neuroscience. Most studies on tridiagonal reaction-diffusion systems assume symmetric diffusion ($b_i = c_i$), which limits applicability to biased transport phenomena.

We study the general non-symmetric case ($b_i \neq c_i$), prove global existence and positivity of solutions via semigroup methods, and provide explicit diagonalizability conditions.

Problem Statement

We consider the following reaction-diffusion system:

$$\begin{cases} \dfrac{\partial U}{\partial t} - A_m \Delta U = F(U) & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial U}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ U(x,0) = U_0(x) & \text{in } \Omega \end{cases}$$

where:

$U = (u_1, \dots, u_m)^T$ is the vector of unknown functions,

$\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary,

A_m is a **non-symmetric tridiagonal matrix** of the form:

$$A_m = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & c_{m-1} & a_m \end{pmatrix}, \quad \text{with } b_i \neq c_i,$$

$F(U) = (f_1(U), \dots, f_m(U))^T$ is a nonlinear reaction term,

∂_η denotes the outward normal derivative (Neumann boundary condition),

U_0 is the initial data.

The main question: Under what conditions on A_m, F, and U_0 does this system admit a unique global solution $U(x,t)$ (defined for all $t > 0$) that remains positive ($u_i(x,t) > 0$ for all i and all $t > 0$)

METHOD

To prove global existence and positivity, we follow the procedure below.

- **Diagonalisation:**
  - Construct diagonal matrix S with $s_1 = 1$, $s_{i+1} = s_i\sqrt{b_i/c_i}$
  - Obtain $SA_mS^{-1} = \tilde{A}_m$ (symmetric positive definite)
  - A_m has real positive eigenvalues $\lambda_s > 0$ and positive eigenvectors $V_s > 0$, $W_s > 0$
- **Semigroup:**
  - Operator $\mathcal{A}U = A_m\Delta U$ generates a positive analytic semigroup
  - $e^{t\mathcal{A}} = S^{-1}e^{t\tilde{A}_m\Delta}S$
- **Construction of an upper bound function:**
  - Let $y(t)$ solve $y'(t) = C(1 + my(t)^p)$, $y(0) = \|U_0\|_\infty$
  - Under $p < 1 + 2/n$, $y(t)$ exists for all $t \geq 0$ (no finite-time blow-up)
  - Define $\overline{U}(x,t) = (y(t), \dots, y(t))^T$
- **Verification that $\overline{U}$ is an upper bound:**
  - $\partial_t\overline{U} = y'(t)\mathbf{1}$, $\Delta\overline{U} = 0$

RESULTS AND DISCUSSION

Main Theorem

Under the conditions:

$$b_i c_i < a_i a_{i+1} \quad \text{and} \quad p < 1 + \frac{2}{n}$$

The reaction-diffusion system admits a **unique global positive solution**:

- **Global existence:** $T_{\max} = \infty$ (solution exists for **all** $t > 0$)
- **Positivity:** $u_i(x,t) > 0$ for all i, x, $t > 0$
- **Uniqueness:** The solution is unique

The symmetry assumption $b_i = c_i$ is **not necessary**.

Key insights:

- First result for non-symmetric systems ($b_i \neq c_i$), generalizing [1] and [2]
- $b_i c_i < a_i a_{i+1}$ ensures A_m is diagonalizable with real positive eigenvalues
- $\|U\|_\infty \leq y(t)$ with $y' = C(1 + my^p)$ prevents finite-time blow-up under $p < 1 + 2/n$
- A_m is Metzler ($b_i, c_i \geq 0$) and F is quasi-positive $\Rightarrow$ the solution remains positive
- Applications include biased diffusion in ecology, chemical reactors, and epidemiology
- The critical case $p = 1 + 2/n$ and the asymptotic behavior remain open

CONCLUSION

We showed that the symmetry assumption $b_i = c_i$ is not required for global existence. Under $b_i c_i < a_i a_{i+1}$ and $p < 1 + 2/n$, the system admits a unique global positive solution. The proof uses symmetrization, analytic semigroup theory, and an explicit upper bound $y(t)$ that prevents blow-up. Positivity is guaranteed by the Metzler property of A_m and quasi-positivity of F. Our results extend previous works to the non-symmetric case.

Bibliography

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FUTURE WORK

Future work includes the study of the critical case $p = 1 + 2/n$, the asymptotic behavior of solutions as $t \to \infty$, and the extension to nonlinear diffusion systems.