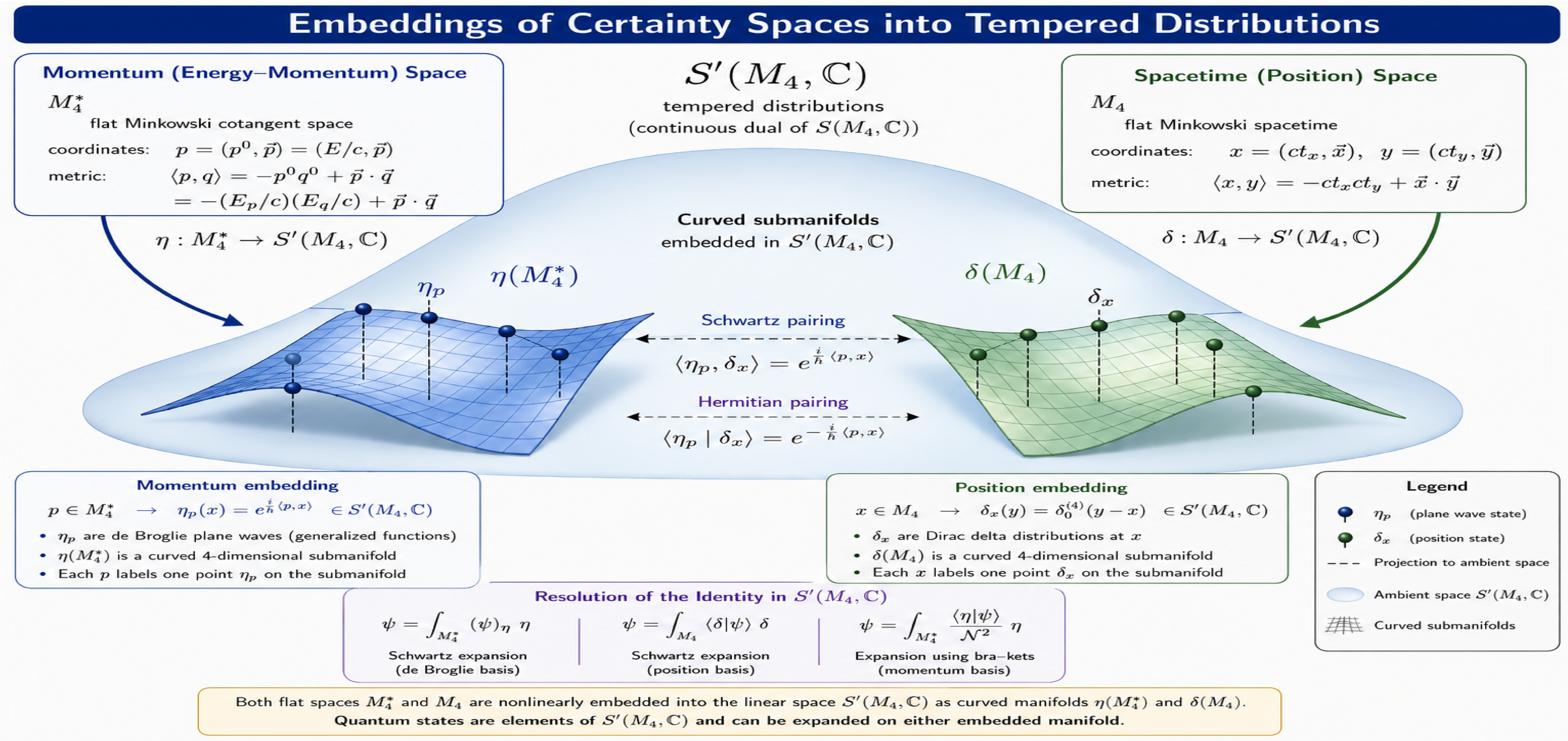
From Hamilton–Jacobi Theory to the Relativistic Schrödinger Picture via von Neumann-like Linear Extension in Tempered Distribution Spaces

David Carfì (david.carfi@unime.it)

University of Messina, Italy University of California Riverside, USA



Aim

We develop a structural bridge between relativistic Hamilton–Jacobi theory and relativistic Schrödinger dynamics within the framework of tempered distributions and Schwartz Linear Algebra. The central question is:

Can the relativistic Schrödinger equation be obtained as a genuine linear extension of Einstein's energy relation?

The answer proposed here is affirmative: the relativistic Schrödinger equation is the Schwartz–von Neumann extension of the classical dispersion law.

Introduction: Hamilton–Jacobi Complete Integral

For a free relativistic particle of rest mass m_0, the Hamiltonian is

$$H_{m_0}(\vec{p}) = c\sqrt{m_0^2 c^2 + |\vec{p}|^2}.$$

For every four-momentum p on the positive mass shell

$$H_{m_0}^+ = \{p \in M_4^* : \ p^0 > 0, \ \langle p, p \rangle = -(m_0 c)^2\},$$

define the principal action

$$S_p(x) = \langle p, x \rangle = \vec{p} \cdot \vec{x} - E_p t.$$

Then

$$\partial_t S_p + H_{m_0}(\nabla_{\vec{x}} S_p) = 0.$$

Thus the family $(S_p)_{p \in H_{m_0}^+}$ is a complete integral of the relativistic Hamilton–Jacobi equation.

Methods: From Actions to de Broglie Waves

The circular exponential map sends each principal action into a de Broglie wave:

$$\eta_p(x) = \exp\left(\frac{i}{\hbar} S_p(x)\right).$$

Hence $p \longmapsto S_p \longmapsto \eta_p$. The map $\eta : M_4^* \to \mathcal{S}'(M_4, \mathbb{C})$ embeds the certainty momentum space into the tempered distribution state space $V = \mathcal{S}'(M_4, \mathbb{C})$. The family $\eta = (\eta_p)_{p \in M_4^*}$ is a Schwartz basis of V, so every tempered state has the expansion

$$\psi = \int_{M_4^*} (\psi)_\eta \, \eta.$$

Central Skeme

Einstein energy relation $E = H_{m_0}(\vec{p})$

↓

Hamilton–Jacobi complete integral $S_p(x) = \langle p, x \rangle$

↓

Circular exponential / de Broglie basis $\eta_p = \exp(iS_p/\hbar)$

↓

Schwartz–von Neumann extension $\bar{f}_\eta(\eta_p) = f(p)\eta_p$

↓

Relativistic Schrödinger equation $i\hbar\partial_t \psi = \bar{H}_{m_0}\psi$

Theory: Schwartz–von Neumann Extension

Let $f : M_4^* \to \mathbb{R}$ be a smooth slowly increasing function. Its Schwartz–von Neumann extension with respect to the de Broglie basis is the unique continuous linear operator

$$\bar{f}_\eta : V \to V$$

defined by

$$\bar{f}_\eta(\eta_p) = f(p)\eta_p.$$

Equivalently,

$$\bar{f}_\eta(\psi) = \int_{M_4^*} f \, (\psi)_\eta \, \eta.$$

Important examples:

$$p_j \longmapsto -i\hbar\partial_j,$$

$$E \longmapsto i\hbar\partial_t,$$

$$H_{m_0} \longmapsto \bar{H}_{m_0}.$$

Result: Main Theorem

The relativistic Schrödinger equation is the Schwartz–von Neumann extension of Einstein's energy equation.

Indeed, Einstein's classical relation

$$E = H_{m_0}(\vec{p})$$

extends, through the de Broglie Schwartz basis, to

$$\bar{E}_\eta = \bar{H}_{m_0}.$$

Since

$$\bar{E}_\eta = i\hbar\partial_t,$$

we obtain

$$i\hbar\partial_t\psi = \bar{H}_{m_0}\psi.$$

Thus quantum dynamics is the continuous $\mathcal{S}$-linear completion of the classical certainty-momentum relation.

Result: Mixed Extension with Scalar Potentials

For a Hamiltonian of separated form

$$H(p, x) = H_{m_0}(p) + V(x),$$

two canonical bases coexist:

η_p for momentum diagonalization,

δ_x for position diagonalization.

The mixed Schwartz–von Neumann extension is

$$\bar{H}_{(\eta,\delta)} = \bar{H}_{m_0,\eta} + \bar{V}_\delta.$$

Therefore,

$$i\hbar\partial_t\psi = (\bar{H}_{m_0} + V)\,\psi.$$

For non-constant potentials, de Broglie waves are no longer eigenmodes of the full Hamiltonian, but the extension mechanism remains exact.

Discussion: Geometric Interpretation

The construction transforms the classical affine geometry of Hamilton–Jacobi generators into the curved geometry of principal waves:

$$M_4^* \longrightarrow S(M_4^*) \longrightarrow \eta(M_4^*) \subset \mathcal{S}'(M_4, \mathbb{C}).$$

Classical side:

$$p \mapsto S_p.$$

Quantum side:

$$p \mapsto \eta_p.$$

Classical elimination of parameters is replaced by Schwartz superposition:

$$\psi = \int_{M_4^*} a(p) \exp\left(\frac{i}{\hbar} S_p\right).$$

In this sense, quantization appears as an infinite-dimensional complex linearization of the classical certainty space.

Application: Maxwell–Schrödinger Transport

For a fixed polarization direction e, let

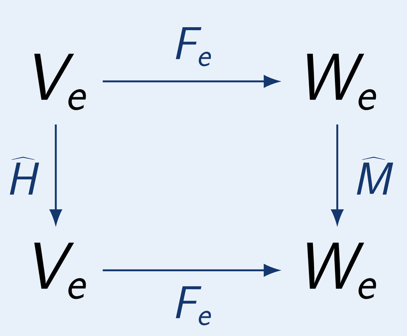
$$F_e : V_e \to W_e$$

be the de Broglie–Maxwell isomorphism defined by

$$F_e(\eta_p) = w_{e,p}.$$

It intertwines scalar and Maxwell-type dynamics:

$$F_e \circ \bar{H} = \bar{M} \circ F_e.$$



Thus dispersion relations, translation representation, and polarization geometry are preserved.

Conclusions

- The Hamilton–Jacobi complete integral generates the de Broglie Schwartz basis.
- Einstein's energy relation extends uniquely to a continuous operator on $\mathcal{S}'(M_4, \mathbb{C})$.
- The relativistic Schrödinger equation is the resulting Schwartz–von Neumann extension.
- Scalar potentials are incorporated through mixed momentum-position extensions.
- The construction transports to Maxwell–Schrödinger fields via de Broglie–Maxwell isomorphisms.

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