Entropy Measures in Finance and Risk Neutral Densities

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Entropy Measures and Finance

- The term entropy was first used in 1865 by German physicist Rudolf Clausius in the field of thermodynamics.
- In Probability theory entropy measures the uncertainty of the random variable.
- The application of entropy in finance can be regarded as the extension of the information entropy and the probability entropy.

Entropy Measures and Finance

- In last two decades the entropy has been important tool for the selection of portfolio, asset pricing and option pricing, Measuring Stock market volatility, in financial literature.

- The Black-Scholes Model (1973) is a Mathematical description of financial markets and derivative investment instruments.

- The present structure of stock option pricing is based on Black and Scholes model, but the model has some restricted assumptions and contradicts with modern research in financial literature.

Entropy Measures and Finance

- In Option Pricing the Black-Scholes model (1973) is crucial and the model is governed by Geometric Brownian Motion (GBM) based on stochastic calculus.

- The current technology of stock option pricing (Black-Scholes model) depends on two factors.
- (a) No arbitrage which implies the universe of risk-neutral probabilities.

- (b) Parameterization of risk-neutral probability by a reasonable stochastic process.

- This implies risk-neutral probabilities are vital in this framework.

- The EPT of Les Gulko is an alternative approach of constructing risk-neutral probabilities without depending on stochastic calculus.
- The work of Les Gulko (1997) "Entropy Pricing Theory" is a fundamental tool for further developments in finance using Statistical Physics.

Some Frequently Used Entropy Measures in Finance

- Some examples of entropy measures mostly used in finance are: Shannon′s, Tsallis,Kaniadakis,Renyi′s, Ubriaco′s, and Shafee′s and many others extensions.

- **The Shannon Entropy**: The Shannon entropy for a discrete time /continuous time random variable of probability measure p on a finite set X is given by respectively:

$$H(P) = -\sum_{i=1}^{n} p_i \ln p_i \text{ where } \sum_{i=i}^{n} p_i = 1,\ p_i \geq 0 \text{ and } 0 \ln 0 = 0$$

$$H(f) = -\int_{-\infty}^{\infty} f(x) \ln f(x)\, dx,\quad f(x) \geq 0$$

Tsallis Entropy

- **Tsallis Entropy**: Tsallis entropy of order q of a probability measure p on a finite set X is defined as :

$$H_q(p) = \begin{cases} \dfrac{1}{q-1}\left[1 - \sum\limits_{i \in X} p_i^q\right] & \text{If } q \neq 1 \\ -\sum\limits_{i \in X} p_i \ln p_i & \text{If } q = 1 \end{cases}$$

Ubriaco Entropy Measure

- Ubriaco (2009) proposed a new entropy measure based on fractional calculus and showed this new entropy has the same properties as Shannon entropy except additivity.

$$H(p) = \sum_{i=1}^{n} p_i \left(\ln\left(\frac{1}{p_i} \right) \right)^d$$

- Similarly in the continuous case:

$$H(\mathrm{g}(x)) = E^g \left(\ln\left(\frac{1}{g(x)} \right) \right)^d, \quad 0 < g \le 1$$

Shafee Entropy Measure

- We obtain the Shannon entropy for d=1

- In 2007 non additive entropy was proposed by **Shafee**. It gives the general form that is non-extensive like Tsallis, but is linearly dependent on component entropies.
- The mathematical description of the new entropy functions was in discrete case and for our problem we consider the continuous cases as an analogue of the discrete case:

$$H_q(g) = -E^g\left[g(x)^{q-1}\ln g(x)\right],\ q>0, q\neq 1$$

Rényi Entropy Measure

- The Rényi entropy is named after Alfréd Rényi and in the field of information theory the Rényi entropy generalizes, the Shannon entropy.
- In financial modeling most of the entropy based calibration methods depend on the use of the logarithmic entropy measure of Shannon.
- A one parameter family of entropies generalizing the logarithmic entropy measure of Shannon and Wiener was considered by Rényi.

Rényi Entropy Measure

- In 2007 Dorje et al. [6] have studied the case of Rényi entropy for calibration of option pricing and explained a piece of information that one drawback in the use of logarithmic entropy measures.
- The use of Rényi entropy is the more general case for the calibration of risk neutral price distribution.
- The Rényi entropy of order r where r is greater than zero, is defined as:

Rényi Entropy Measure

- i.e.

$$H_r(X) = \frac{1}{1-r} \log\left(\sum_{i=1}^{n} g_i^r(X)\right)$$

- In the case of continuous random variable we may write Rényi entropy as:

$$H_r(X) = \frac{1}{1-r} \log E^g\left(g^{r-1}(X)\right)$$

- the Rényi entropy is just the logarithm of the size of the support of and for r approaches to 1 , equals to Shannon's entropy.

EPT Framework

- *Consider a risky asset on time interval* $[0,T]$.
- *Let* Y_T *be asset price process of* S_T *at future time* T.
- G *as state space, a subset of real line* R, $g\left(S_T\right)$ *the probability densities on* P
- $H\left(g\right)$ *the index of market uncertainty about* Y_T.
- *The* $H\left(g\right)$ *is defined on the set of beliefs* $\mathrm{g}\left(S_T\right)$

therefore the efficient market belief $f\left(S_T\right)$ *maximizes* $H\left(g\right)$

EPT Framework

- Also,

- *The index of the market uncertainty about* Y_T *as a Shannon entropy can be written as*:

$$H(g) = -E^g\left[\ln\ g(Y_T)\right]$$

- $f(S_T)$ *which maximize* $H(g)$ *is called the entropy of random variable* Y_T *and used to measure the degree of uncertainty of* $g(S_T)$.

- *The maximum entropy market belief* $f(S_T)$ *as a solution to the maximum entropy problem can be written as follows*:

$$f = \arg\ \max\ \{H(g), g \in G\}$$

Shafee Entropy and Risk Neutral Densities

- Preda and Sheraz have recently studied Shafee entropy maximization problems for the case of risk neutral densities. They have obtained solutions for various cases such as weighted Shafee entropy maximization problem, Expected Utility-Weighted Shafee Entropy (EU-WSE) frame work .

- The weighted entropy was first defined by Guiasu [11], considering the two basic concepts of objective probability and subjective utility.

Shafee Entropy and Risk Neutral Densities

- **Theorem 2.1.**

$$\max -E^{g}\left[u(Y_T)g^{q-1}(Y_T)\ln \mathrm{g}(Y_T)\right]$$

subject to

$$E^{g}\left[I_{\{Y_T>0\}}\right]=1 \qquad \text{C-1}$$

$$E^{g}\left[u(Y_T)\varphi_i(Y_T)\right]=a_i \;,i=\overline{1,n}, \qquad \text{C-2}$$

where $u(S_T)>0,\varphi_i: R\to R$ *and* $a_1,...a_n$ *are given real values.The solution of the problem is*

$$\mathrm{g}(S_T)=\left[\frac{qu(S_T)W\left(\frac{\left(\alpha+\sum_{i=1}^{n}\beta_i u(S_T)\varphi_i(S_T)\right)(1-q)}{qu(S_T)}\exp\left\{-\left(\frac{1-q}{q}\right)\right\}\right)}{\left(\alpha+\sum_{i=1}^{n}\beta_i u(S_T)\varphi_i(S_T)\right)(1-q)}\right]^{\frac{1}{1-q}}.$$

where $\alpha,\beta_1,...\beta_n$*are Langrage multipliers, and* W *is the Lambert function.*

Ubriaco Entropy Measure and Risk Neutral Densities

- We consider the analogue of the discrete case of non additive entropy defined by Ubriaco. We suppose that all expectations are also well defined and underlying optimization problems admit solutions for some continuous cases.

Lemma $\psi = x\left(\ln\frac{1}{x}\right)^{d}, 0 < x \leq 1, d > 0, \ \psi' : \left(0, e^{-d}\right) \to \left(0, \infty\right)$. *Then* ψ' *the first derivative of* ψ *admits inverse.*

Ubriaco Entropy Measure and Risk Neutral Densities

- Consider the following example of theorem for risk neutral density via Ubriaco entropy measure.
- **Theorem 2.2**

$$\max H(g) = E^{g}\left[\left(\ln\left(\frac{1}{g(Y_T)}\right)\right)^{d}\right]$$

subject to

$$E^{g}\left[I_{\{Y_T>0\}}\right]=1 \qquad \text{C-1}$$

$$E^{g}\left[Y_T\right]=S_0e^{rT} \qquad \text{C-2}$$

$$E^{g}\left[\left(Y_T-K_0\right)^{+}\right]=C_0e^{rT} \qquad \text{C-3}$$

where K_0 is the strike price, T is time to expiry and r is the risk free interest rate.
Then the risk neutral density is given by:

Ubriaco Entropy Measure and Risk Neutral Densities

$$g\left(S_T\right)=\left(\psi'\right)^{-1}\left(\lambda+\beta_1 S_T+\beta_2\left(S_T-K_0\right)^+\right)$$

where λ,β_1 *and* β *are Lagrange multipliers and need to be determined by* $C-1, C-2$ *and* $C-3$.

We can extend the previous result of theorem for the case of weighted entropy. The weighted entropy was first defined by Guiasu [11], *considering the two basic concepts of objective probability and subjective utility. If* $u\left(S_T\right)>0$ *then solution of the weighted Ubriaco entropy maximization problem subject to the constraints* $C-1, C-2$ *and* $C-3$ *is given by*:

$$g\left(S_T\right)=\frac{\left(\psi'\right)^{-1}\left(\lambda_1+\beta_1 S_T+\beta_2\left(S_T-K_0\right)^+\right)}{u\left(S_T\right)}$$

Rényi Entropy Measure and Risk Neutral Densities

- **Theorem 2.3.** Consider the Rényi entropy maximization problem:

$$\max H_r(g) = \frac{1}{1-r} \ln E^g\left(g^{r-1}(Y_T)\right)$$

subject to

$$E^g\left[I_{\{-\infty<Y_T<\infty\}}\right] = 1 \qquad \text{C-1}$$

$$E^g\left[\varphi_i(Y_T)\right] = c_i \ , i = \overline{1,n}, \qquad \text{C-2}$$

where $,\varphi_i : \Lambda \to R$ *and* $c_1, \ldots, c_n$ *are given real values.* Λ *is the state space of the prices.*

The solution of the problem is given by:

$$g(S_T) = \frac{\left[1 - \frac{1-r}{r}\left(\sum_{i=1}^{n} \beta_i \left(c_i - \varphi_i(S_T)\right)\right)\right]^{\frac{1}{r-1}}}{\int_{-\infty}^{+\infty} \left[1 - \frac{1-r}{r}\left(\sum_{i=1}^{n} \beta_i \left(c_i - \varphi_i(S_T)\right)\right)\right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1, \ldots, \beta_n$ *are to be obtained by* C-1 *and* C-2.

Weighted-Rényi Entropy Measure and Risk Neutral Densities

Theorem 2.4 Consider the weighted- Rényi entropy maximization problem:

$$\max H_r(g) = \frac{1}{1-r} \ln E^g\left(u(Y_T)\, g^{r-1}(Y_T)\right)$$

subject to

$$E^g\left[I_{\{-\infty < Y_T < \infty\}}\right] = 1 \qquad \text{C-1}$$

$$E^g\left[\varphi_i(Y_T)\right] = c_i \ , i = \overline{1,n}, \qquad \text{C-2}$$

In particular $\varphi_i(S_T) = u(S_T)$, $c_1 = 1$ *then* $E^g\left(u(S_T)\right) = 1$, $\varphi_i : \Lambda \to R$ *and* $c_1, \ldots, c_n$ *are given real values.*

Λ *is the state space of prices and* u *is a weighted function then solution of the problem is*

$$g(S_T) = \frac{\left[\dfrac{1 - \dfrac{1-r}{r}\left(\sum_{i=1}^{n} \beta_i\left(c_i - \varphi_i(S_T)\right)\right)}{u(S_T)}\right]^{\frac{1}{r-1}}}{\displaystyle\int_{-\infty}^{+\infty} \left[\dfrac{1 - \dfrac{1-r}{r}\left(\sum_{i=1}^{n} \beta_i\left(c_i - \varphi_i(S_T)\right)\right)}{u(S_T)}\right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1, \ldots, \beta_n$ *are to be obtained by* C-1 *and* C-2.

Rényi Entropy Measure and EU-WE Framework

- Casquilho et al. [4] used EU-WE i.e. expected utility-weighted entropy framework under a 1-parameter generalization of Shannon formula focused on an ecological and economic application at the landscape level. Following this approach the EU-WE framework, if u is a positive utility application on G . In the next theorem we present a new result for the risk-neutral density using the frame of expected utility-weighted entropy for the case of Rényi-entropy maximization problem.

- **Theorem 2.5** Consider the case of Rényi-entropy maximization problem:

$$\max H_r(g) = E^g\left(u(Y_T)\right) + \frac{1}{1-r}\ln E^g\left(u(Y_T)g^{r-1}(Y_T)\right) - \ln E^g\left(u(Y_T)\right)$$

subject to the constraints $C-1$ *and* $C-2$ *of theorem* 2.4. *Then solution of the problem is given by:*

$$\mathrm{g}(S_T) = \frac{\left[\frac{1}{r\left(1+\sum_{i=1}^{n}\beta_i+\lambda\right)} - \frac{1-r}{r} + \frac{1-r}{r}\frac{\left(\lambda+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)}{u(S_T)}\right]^{\frac{1}{r-1}}}{\int_{-\infty}^{+\infty}\left[\frac{1}{r\left(1+\sum_{i=1}^{n}\beta_i+\lambda\right)} - \frac{1-r}{r} + \frac{1-r}{r}\frac{\left(\lambda+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)}{u(S_T)}\right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1,\dots,\beta_n$ *are to be obtained.*

Pricing European Call and Put Options

- We use the new risk neutral density functionS obtained in the case of Rényi entropy, to evaluate the European call and put options on a dividend protected stocks. At time to expiry T, a call option pays $\max(0, K - S_T)$ and a put option pays $\max(0, S_T - K)$ is the K strike price.

- We use the linear pricing rule and the risk neutral density . Then the price of European Call can be written as follows:

Pricing European Call and Put Options

- For European Call:

$$Call = PE^{g}\left(\max\left(0, S_T - K\right)\right)$$

$$= K\int_{K}^{\infty} g\left(S_T\right) S_T dS_T - PK \int_{K}^{\infty} g\left(S_T\right) dS_T$$

$$Call = Pc_1 - PKG\left(K\right) - PK + PK\left(g\left(K\right) - 0\right) + P\int_{0}^{K} G\left(S_T\right) dS_T$$

- For European Put:

$$Put = PE^{g}\left(\max\left(0, K - S_T\right)\right)$$

$$Put = P\int_{0}^{K} G\left(S_T\right) dS_T$$

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