Serial Risk Sharing

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Introduction and motivation

- Peer-to-peer (P2P) risk sharing redistributes losses among agents in a pool:

$$\underset{\text{(before)}}{(X_1,\dots,X_n)} \longrightarrow \underset{\text{(after)}}{(Y_1,\dots,Y_n)}$$

- **Uniform risk sharing:**

$$Y_i^{\mathrm{U}} = \frac{S}{n}, \qquad S = \sum_{i=1}^{n} X_i$$

  ▷ 👍 simple

  ▷ 👎 may be less suitable for heterogeneous losses

- **Conditional mean risk sharing (CMRS):**

$$Y_i^{\mathrm{CMRS}} = \mathbb{E}[X_i|S], \qquad S = \sum_{i=1}^{n} X_i,$$

  ▷ 👍 attractive theoretical properties

  ▷ 👎 often computationally difficult in practice

- We propose **serial risk sharing (SRS):** a transparent, closed-form risk-sharing rule tailored for heterogeneous losses.

  ▷ 💡 inspired by *serial cost sharing* (Moulin and Shenker, 1992, *Econometrica 60*(5))

  ▷ **decompose-then-allocate** approach: (i) decomposing each loss into quantile-based layers and (ii) allocating each layer among relevant agents

Step 1: Decompose

- F_i: CDF of agent i's initial loss X_i (assumed to be continuous)
- $U_i := F_i(X_i) \sim \mathrm{U}(0,1)$ by probability transformation

  ▷ larger U_i ➔ agent i is *more unlucky* relative to their own distribution

- **Quantile equivalent loss for agent j:** $\widetilde{X}_{j,i} = F_j^{-1}(U_i) = F_j^{-1}(F_i(X_i))$

  ▷ $F_j^{-1}(u) = \inf\{x \in \mathbb{R} : F_j(x) \geq u\}$: quantile function of X_j

  ▷ **Interpretation:** the loss agent j would have if they *were* as unlucky as agent i.

  ▷ It translates agent i's realized loss severity into agent j's loss scale.

  ▷ Agent j is "responsible" for this loss amount when considering the allocation of X_i.

  ▷ $\widetilde{X}_{j,i} \overset{\mathrm{d}}{=} X_j$ by quantile transformation.

$$X_i \xrightarrow{\text{probability scale}} U_i = F_i(X_i) \xrightarrow{\text{agent } j\text{'s scale}} \widetilde{X}_{j,i} = F_j^{-1}(U_i)$$

- **Sorted indices:** $(1)_i, \dots, (n)_i$ are sorted from $1, \dots, n$ such that quantiles at U_i are ordered:

$$F_{(1)_i}^{-1}(U_i) \leq \cdots \leq F_{(n)_i}^{-1}(U_i) \quad \text{or} \quad \widetilde{X}_{(1)_i,i} \leq \cdots \leq \widetilde{X}_{(n)_i,i}$$

  ▷ We let $\widetilde{X}_{(j)_i,i} := F_{(j)_i}^{-1}(U_i)$ with $\widetilde{X}_{(0)_i,i} := 0$.

- **Decomposition:**

$$X_i \longrightarrow \underset{\text{(quantile layers)}}{\widetilde{X}_{(j)_i,i} - \widetilde{X}_{(j-1)_i,i}}$$

Step 2: Allocate

- Allocation of each quantile layer $\widetilde{X}_{(j)_i,i} - \widetilde{X}_{(j-1)_i,i}$ to agent k:

  ▷ **Availability:** the layer should be "contained" in agent i's loss ($\widetilde{X}_{(j)_i,i} \leq X_i$)

  ➔ only layers up to the full amount X_i are to be allocated

  ▷ **Relevance:** the layer should be "contained" in agent k's quantile equivalent loss ($\widetilde{X}_{(j)_i,i} \leq \widetilde{X}_{k,i}$)

  ➔ agent k only gets assigned layers "within their responsibility"

  ▷ For positive quantile layer $\widetilde{X}_{(j)_i,i} - \widetilde{X}_{(j-1)_i,i}$ satisfying *availability* condition, the *relevance* condition is fulfilled for exactly $n-j+1$ agents

  ➔ natural to divide the layer equally among them

- **Definition of SRS:** summing the allocated amounts to agent k from $X_1, \dots, X_n$ yields:

$$Y_k^{\mathrm{S}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\widetilde{X}_{(j)_i,i} - \widetilde{X}_{(j-1)_i,i}}{n-j+1} \mathbf{1}_{\{\widetilde{X}_{(j)_i,i} \leq X_i \wedge \widetilde{X}_{k,i}\}} \quad \text{for } k = 1, \dots, n.$$

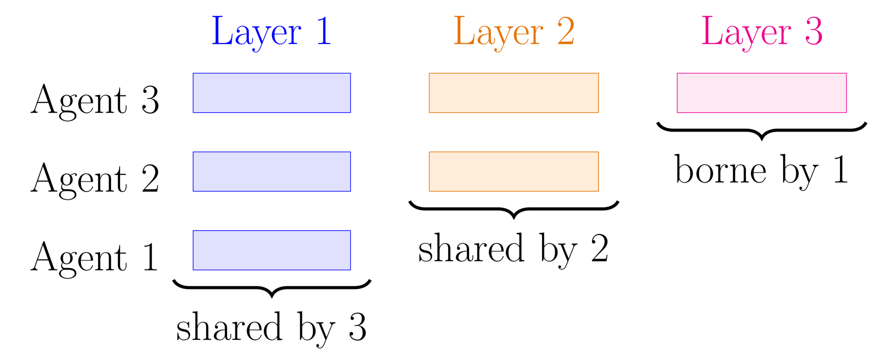
Ordered case

- **First-order stochastic dominance:** $X \leq_{\mathrm{st}} Y$ if $F_X^{-1}(u) \leq F_Y^{-1}(u)$ for $u \in (0,1)$, where F_X^{-1} and F_Y^{-1} are quantile functions of X and Y, respectively.
- **Simplification of SRS under ordering:** If $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$, then SRS simplifies to

$$Y_k^{\mathrm{S}} = \sum_{j=1}^{k}\sum_{i=j}^{n} \frac{\widetilde{X}_{j,i} - \widetilde{X}_{j-1,i}}{n-j+1} \quad \text{for } k = 1, \dots, n.$$

  ▷ SRS becomes a sequence of uniform risk sharing applied layer by layer:

$$Y_1^{\mathrm{S}} = \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{X}_{1,i} - \widetilde{X}_{0,i}\right), \quad Y_2^{\mathrm{S}} = \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{X}_{1,i} - \widetilde{X}_{0,i}\right) + \frac{1}{n-1}\sum_{i=2}^{n}\left(\widetilde{X}_{2,i} - \widetilde{X}_{1,i}\right), \dots$$



Main theoretical results

- **Full allocation:** $\sum_{i=1}^{n} Y_i^{\mathrm{S}} = \sum_{i=1}^{n} X_i$.
- **Actuarial fairness:** $\mathbb{E}[Y_i^{\mathrm{S}}] = \mathbb{E}[X_i]$ for $i = 1, \dots, n$.
- **Monotonicity:** if $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$, then $Y_1^{\mathrm{S}} \leq \cdots \leq Y_n^{\mathrm{S}}$.
- **Scale invariance:** $X_i \mapsto cX_i,\ i = 1, \dots, n$ implies $Y_i^{\mathrm{S}} \mapsto cY_i^{\mathrm{S}},\ i = 1, \dots, n$ for $c > 0$.

Additional stochastic orders

▷ **Convex order:** $X \leq_{\mathrm{cx}} Y$ if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ for all convex functions φ such that both expectations are finite.

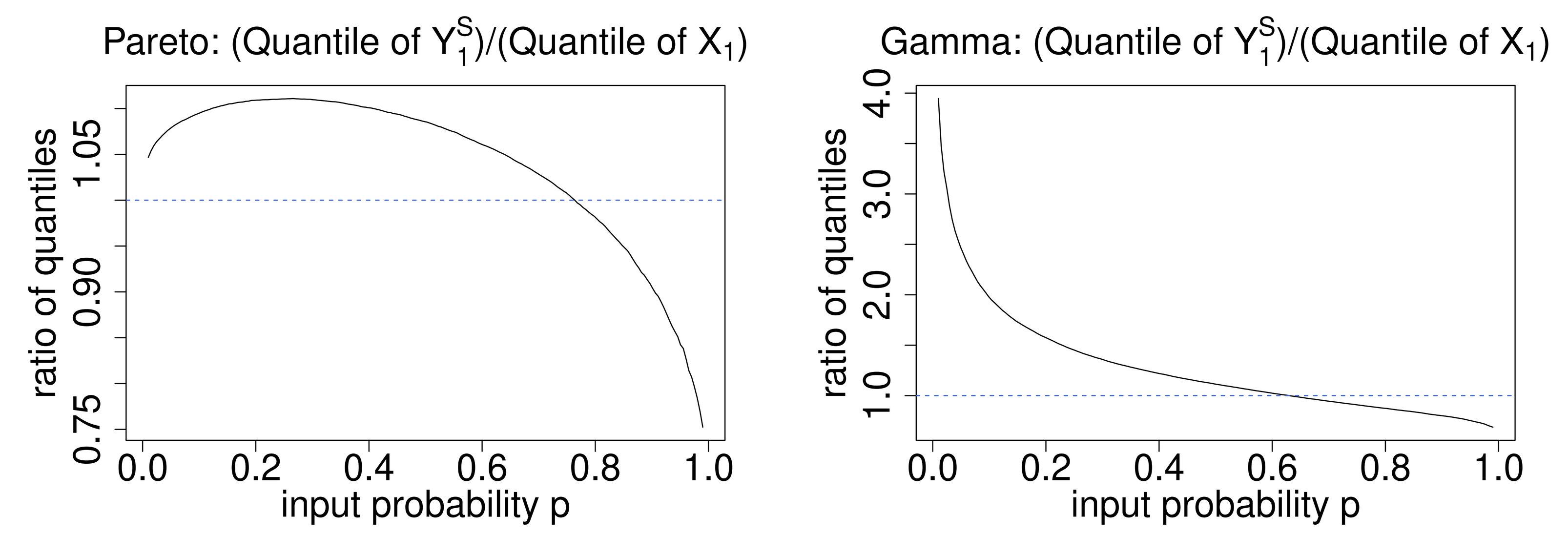
▷ **Dispersive order:** $X \leq_{\mathrm{disp}} Y$ if $F_Y^{-1}(u) - F_X^{-1}(u)$ is increasing in u on $(0,1)$.

The following are under the condition that $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$ and $X_1 \leq_{\mathrm{disp}} \cdots \leq_{\mathrm{disp}} X_n$.

- **Risk fairness:** $Y_i^{\mathrm{S}} \leq \operatorname{ess\,sup} X_i$ for $i = 1, \dots, n$, where $\operatorname{ess\,sup} X_i := \inf\{x \in \mathbb{R} : \mathbb{P}(X_i > x) = 0\}$ denotes the essential supremum of X_i.
- **Convex-order improvement:** $Y_i^{\mathrm{S}} \leq_{\mathrm{cx}} X_i$ for $i = 1, \dots, n$.
- **Risk-measure reduction:** $\rho(Y_i^{\mathrm{S}}) \leq \rho(X_i)$ for $i = 1, \dots, n$ and risk measure ρ satisfying subadditivity, comonotonic additivity, positive homogeneity, and law invariance.

Numerical findings

- **Effects on quantiles after SRS:**



Ratios of post-SRS to pre-SRS quantiles for agent 1. Left: $X_1 \sim \mathrm{Pareto}(3,1)$, $X_2 \sim \mathrm{Pareto}(4,1.1)$, *and* $X_3 \sim \mathrm{Pareto}(5,1.2)$. *Right:* $X_1 \sim \mathrm{Gamma}(2,2)$, $X_2 \sim \mathrm{Gamma}(3,3)$, *and* $X_3 \sim \mathrm{Gamma}(4,4)$. $\mathrm{Pareto}(\alpha,\theta)$: *Pareto distribution with shape parameter* $\alpha > 0$ *and scale parameter* $\theta > 0$. $\mathrm{Gamma}(\alpha,\lambda)$: *gamma distribution with shape parameter* $\alpha > 0$ *and rate parameter* $\lambda > 0$.

▷ Tail quantile ratios fall below 1, suggesting mitigation of extreme losses, from the reduction in loss quantiles at high probability levels (associated with tail events).

Conclusion

- SRS is a transparent, closed-form P2P risk-sharing rule for heterogeneous losses.
- It is based on a decompose-then-allocate mechanism: forming quantile-based layers and then sharing available layers among relevant agents.
- It is fully allocating, actuarially fair, scale invariant, and yields improvements under first-order stochastic and dispersive orderings.
- Numerical results demonstrate that SRS can also help manage tail risk.

Takeaway: SRS serves as an alternative P2P risk-sharing rule, especially applicable to settings where transparency and implementability are important.

References

Boonen, T. J. and K. L. Chiu (2026). Serial risk sharing. *Available at SSRN 5373316.*

Moulin, H. and S. Shenker (1992). Serial cost sharing. *Econometrica 60*(5), 1009–1037.