Modified operators are not sufficient to determine a minimum length scale in quantum gravity

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Summary of Results

- Various formulations of quantum gravity predict the existence of a positive minimum length scale, usually described by a positive lower bound $\Delta x_0 > 0$ on the uncertainty Δx of all possible wave functions.
- Formulation of this follows Kempf, Mangano, and Mann (1995) who modified the position and momentum operators to obtain the modified commutation relation

$$[\hat{x}', \hat{p}'] = [\hat{x}', \hat{p}] = i(1 + \beta p^2) \tag{1}$$

  which has a minimum length scale $\Delta x_0 = 2\sqrt{\beta}$.
- This lead to the idea that all that is needed to have a minimal length scale was for the modified operators to have the commutation relation above.
- **We show that there are operators which satisfy the modified commutation relation but do not have a minimum length scale.**

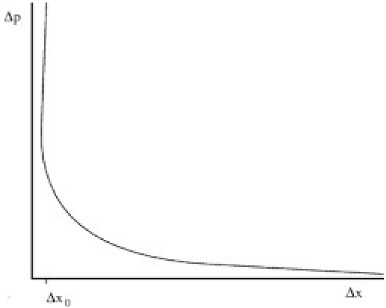
Minimum Length Scale from Quantum Gravity

- Heisenberg Uncertainty Principle (HUP)

  $\Delta x \Delta p \propto 1 \quad \rightarrow \quad \Delta x \propto \frac{1}{\Delta p}$

- Generalized Uncertainty Principle (GUP)

  $\Delta x \Delta p \propto 1 + \beta (\Delta p)^2 \quad \rightarrow \quad \Delta x \propto \frac{1}{\Delta p} + \beta \Delta p$



Modified Commutation Relationship and Uncertainty Relation

- $\Delta x \Delta p \geq \frac{\hbar}{2} \quad \rightarrow \quad \Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right)$

- Using $\Delta A \Delta B \geq \frac{1}{2} \mid \langle [\hat{A}, \hat{B}] \rangle \mid$ and $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$,

  we modify commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \quad \rightarrow \quad i\hbar\left(1 + \beta p^2\right) ,$$

  so that
$$\begin{aligned} \frac{1}{2} \mid \langle [\hat{x}, \hat{p}] \rangle \mid \; &= \; \frac{\hbar}{2}\left(1 + \beta \langle \hat{p}^2 \rangle\right) \\ &= \; \frac{\hbar}{2}\left(1 + \beta (\Delta p)^2 + \beta \langle \hat{p} \rangle^2\right) \end{aligned}$$

Modified commutators imply modified X and P

Different pairs of the position and momentum operators can give the same commutation relation, $[\hat{x}, \hat{p}] = i\hbar\left(1 + \beta p^2\right)$

- Modified x, and unmodified p:

  $\hat{x} = i\hbar(1 + \beta p^2)\partial_p \quad , \quad \hat{p} = p$

- Unmodified x and modified p:

  $\hat{x} = i\hbar\partial_p \quad , \quad \hat{p} = p + \frac{\beta}{3}p^3$

- Modified x and modified p :

  $\hat{x} = i\hbar e^{-\beta p^2/2}\partial_p \quad , \quad \hat{p} = e^{\beta p^2/2}p$

Why Modified Operators Matter?

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right)$$

The minimum length scale will be determined by the modified commutation relation if the all the modified operators result in the above uncertainty relation.

However, there is a notation issue for the uncertainty relation.

Assuming that modified momentum is $\hat{p}' = g(p)$,

- For $1 + \beta\left(\Delta p\right)^2$ (The right hand side of the inequality),

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

- However, for $\Delta x \Delta p$ (The left hand side of the inequality),

$$\Delta p' = \sqrt{\langle g(p)^2 \rangle - \langle g(p) \rangle^2}$$

Uncertainty Relations Are Different

We have to distinguish the modified operators and unmodified operators. Putting prime(') on the modified operators, we get different uncertainty relations for the three cases.

- $\Delta x' \Delta p \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right)$ Modified x ; Unmodified p
- $\Delta x \Delta p' \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right)$ Unmodified x ; Modified p
- $\Delta x' \Delta p' \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right)$ Both x and p Modified

We no longer obtain the form $\frac{1}{\Delta p} + \Delta p$ by cancellation if the momentum operator is modified.

Thus, we need to investigate the existence of the minimum length scale for the uncertainty relation having modified momentum.

Investigation Using Test Function

Since taking uncertainty of an operator is not a linear operation, *i.e.* $\Delta g(p) \neq g(\Delta p)$, we chose an arbitrary test function to investigate the minimum length scale.

The GUP should give a global minimum length for every valid state.

- Test Function: $\Psi(p) = Ce^{-p^2/2\sigma^2}$

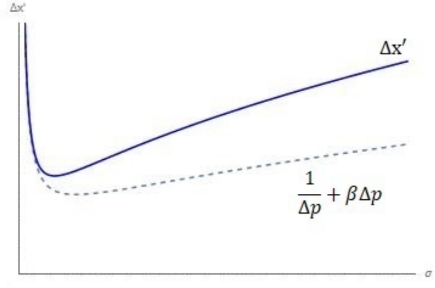
- Modified inner product: $\int_{-\infty}^{\infty} \frac{dp}{f(p)} \Psi(p)^* \Psi(p)$,

  where $\hat{x}' = i\hbar f(p)\partial_p$, so that $(\langle\Psi|\hat{x}')|\Phi\rangle = \langle\Psi|(\hat{x}'|\Phi\rangle)$

- Different choice of $\hat{x}'$ will give different normalization factor C.

Modified X and Unmodified P

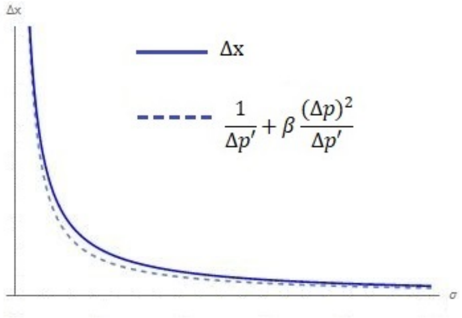
- $\hat{x}' = i\hbar(1+\beta p^2)\partial_p \quad , \quad \hat{p} = p$
- $\Delta x' \Delta p \geq \frac{\hbar}{2}\left(1+\beta\left(\Delta p\right)^2\right) \rightarrow \Delta x' \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p}+\beta \Delta p\right)$ GUP
- $\Delta x' \propto \left(\left(\frac{1}{\sigma}+\frac{3}{2}\beta\sigma\right)e^{-\left(\frac{1}{\sigma^2\beta}\right)}\frac{1}{\int_{\frac{1}{\sqrt{\sigma^2\beta}}}^{\infty} e^{-t^2}dt}\right)^{\frac{1}{2}}$ Direct



Unmodified X and Modified P

- $\hat{x} = i\hbar\partial_p \quad , \quad \hat{p}' = p + \frac{\beta}{3}p^3$
- $\Delta x \Delta p' \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right) \rightarrow$

  $\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p'} + \beta\frac{(\Delta p)^2}{\Delta p'}\right) = \frac{1+\beta\frac{\sigma^2}{2}}{\sqrt{\frac{1}{2}\sigma^2+\frac{\beta}{2}\sigma^4+\frac{5\beta^2}{3}\sigma^6}}$ GUP
- $\Delta x = \frac{\hbar}{\sqrt{2}\sigma}$ Direct



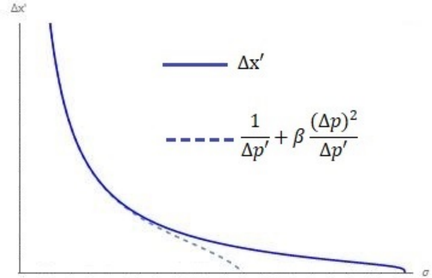
Both X and P Modified

- $\hat{x}' = i\hbar e^{-\beta p^2/2}\partial_p \quad , \quad \hat{p}' = e^{\beta p^2/2} p$

- $\Delta x' \Delta p' \geq \frac{\hbar}{2}\left(1 + \beta\left(\Delta p\right)^2\right) \quad \rightarrow$

  $\Delta x' \geq \frac{1+\beta(\Delta p)^2}{\Delta p'} \propto \frac{1}{\sigma^2}\left(\frac{1}{\sigma^2} - \frac{3\beta}{2}\right)^{3/4}\left(\frac{1}{\sigma^2} - \frac{\beta}{2}\right)^{-5/4}$ GUP

- $\Delta x' \propto \frac{1}{\sigma^2}\left(\frac{1}{\sigma^2} - \frac{\beta}{2}\right)^{1/4}\left(\frac{1}{\sigma^2} + \frac{\beta}{2}\right)^{-3/4}$ Direct



Conclusion

- The modified operators determine the existence of the minimum length scale rather than the modified commutation relationship.
- The position operator should be modified to have the minimum length scale.
- If modified momentum grows faster than the commutation relation, the minimum length is not guaranteed from GUP.