From Calculus of variations to Fractional Euler–Lagrange equations. †

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Abstract:

The role of *calculus of variations* in aerodynamics studies is related to the study of optimization problems in aircraft dynamics which include the study of the optimal trajectory using the least fuel consumption or optimum geometry in aircraft design or optimal distances of the aircraft or rocket trajectory subjects to certain conditions. In the calculus of variations, the optimal solution is obtained by the *Euler-Lagrange equation*. The fractional calculus is a mathematical theory for calculating derivatives and integrals where the order can be fractions or complex numbers. There are numerous applications of this theory in various fields of science and engineering today. In this paper, we have applied theory of *fractional calculus* to formulate the generalized term of *Euler-Lagrange equation*. The ubiquitous equation in mechanics and other field.

Keywords: Fractional Calculus, Euler-Lagrange equation, Tautochrone problem

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1. Introduction

Mathematical theory of calculus plays a major role in describing the physical behavior of various systems in nature. The theory has two aspects to study, the differential calculus and the integral calculus. Traditionally studied calculus are in the form of an integer order. A mathematical theory that studied non-integer order operators are *fractional calculus*. The concept of *fractional calculus* can be traced back to the same period as classical calculus. The theory has originated from the query from the French mathematician *L'Hôpital* asked the German mathematician *Leibniz* about the meaning of half-order derivative in 1695 [1-4]. Although theories evolved around the same period of classical calculus, they were not widely studied because of the unfamiliarity [1].

The theory was interested again in 1974 with the publication by *Oldham and Spanier* [1] and the introduction of new ideas in applications such as the geometric interpretation of operators. In 2002 *Igor Podlubny* proposed an interpretation of fractional integrations as a shadow cast on a wall [5] and in 2003 *Tenreiro Machado* published an interpretation of fractional integrations by describing as probability based on the *Grunwald Letnikov* definition [6]. Today, theory of *fractional calculus* has been widely applied studies in the field of science and engineering include *viscoelasticity, electrochemistry, biological population models*, *nonlinear dynamics, control engineer* and *signals processing* [1-4].

Due to the advanced development of the theory. It was found that the many systems can be described closer to the physical characteristics than the integer-order calculus [1][2]. The fractional derivatives are feasible to describe the memory and hereditary properties of various materials and processes [4]. There are several definitions for fractional integral have been proposed, for example, the fractional integrals of *Riemann-Liouville* and *Liouville*, *Weyl* and fractional derivative also has several definitions such as *Riemann–Liouville*, *Grünwald–Letnikov*, *Caputo*, *Liouville*, *Weyl*, *Riesz*, *Hifler* and *Marchaud* [1-4]. The *Riemann-Liouville* and *Grunwald-Letnikov* are mostly used definitions. The *Riemann-Liouville* definition is suitable for finding the analytical solution; while the *Grunwald-Letnikov* definition is suitable for numerical evaluations [4].

2. Calculus of variations

The branch of mathematical analysis that uses small changes in functions to find the extremum of functional is called *calculus of variations.* The earliest problems posed in the *calculus of variations* is *brachistochrone problem* [10-13]. The least-time variation problem, which was first solved by *Johann Bernoulli* in 1696. The problem of finding the shortest distance between two points in space influence of gravity, which feasible to explain by analytical techniques of *calculus of variations*. The problem described the trajectory of the sliding bead from rest and accelerate by gravity without friction from one point to another in the least time. The solutions of the *calculus variation problems* are able to solving by the second-order partial differential equation of *Euler-Lagrange equation* [10-13].

The applications of *calculus of variations* in the science and engineering fields include solutions to the *brachistochrone problem, tautochrone problem, catenary problem,* geometric optics and *Newton's minimal resistance problem, Plateau's problem* which show the existence of a minimal surface with a given boundary, optimum shapes of aircraft and missile component in aerodynamics which required desirable aerodynamic characteristic such as minimum drag or maximum lift-to-drag ratio and finite element method with a variational method for finding numerical solutions to boundary-value problems in differential equations [9-13];

The concept of the infinitesimal change in *calculus of variations* can be applied with *fractional calculus* operators to provide some useful results. In this paper, we have applied theory of *fractional calculus* to formulate the generalized term of *Euler-Lagrange equation*. The equation mostly uses in mechanics and other fields. The result is obtained by using the *fractional variation principles* in this study, the *fractional Euler-Lagrange equation* is derived.

3. Basic Definitions and Preliminary Results

The *fractional calculus* is the mathematical theory that have many definitions. In this section the definitions of *Riemann and Liouville* fractional derivatives adopt from the book by *Samko* et al., is introduced [3].

Definition 1 The *Euler's Gamma function* is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \tag{1}$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} \tag{2}$$

Definition 2 Let *f* be a real function defined on [0, 1] and $\alpha, \beta > 0$, then the left and right *Riemann-Liouville fractional derivative* of order α, β of the function *f* are defined by the following

The Left Riemann-Liouville Fractional Derivative

$${}_a\mathbf{D}_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{1+\alpha-n}}\, dt, \qquad x > a \tag{3}$$

The Right Riemann-Liouville Fractional Derivative

$$ {}_{x}\mathbf{D}_{b}^{\beta} f(x) = \frac{(-1)^{n}}{\Gamma(n-\beta)} \frac{d^{n}}{dx^{n}} \int_{x}^{b} \frac{f(t)}{(t-x)^{1+\beta-n}} dt, \qquad x < b \tag{4} $$

where $\Gamma(\cdot)$ it is the *Euler Gamma function* and $0 < \alpha, \beta < 1$

4. Euler–Lagrange equation

The theory of physics which explained the motion of macroscopic objects by *Newtonian Mechanics* and *Lagrangian mechanics* is called *Classical Mechanics*. It was found that both theories produced similar results. The *Lagrangian mechanics* are explained in terms of integer-order derivative and generalized coordinates, hence *The Lagrangian mechanics* have much advantage than the *Newtonian mechanics* that more specific to calculations in the orthogonal coordinate system [10-13].

Euler and *Lagrange* have developed the *Euler-Lagrange equation* with their studies on the *tautochrone* problem in the 1750s [9]. The problem of finding the trajectory of the weighted particle fall off to a fixed point in a fixed amount of time, independent of the starting point [10-13]. It is a second-order partial differential equation whose solutions are the stationary functions [9].

$$ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = 0 \tag{5} $$

Theory of fractional calculus, provided the generalized form of the *Euler-Lagrange equation.* Many authors have studied different approached in formulated the *fractional Euler-Lagrange* equation. *Agrawal* has studied the *linear Euler–Lagrange equations* and the transversality conditions for *fractional variational problems* [14]. *Guezane-Lakound* et al., explained the solutions for a nonlinear *fractional Euler-Lagrange* type equation involving both the left *Riemann-Liouville* and the right *Caputo types fractional derivatives* [15]. *Matheus* et al., have obtained the *Euler-Lagrange equation* in integral form by generalizing the *DuBois-Reymond fundamental lemma* to *variational functionals* with *Caputo derivatives* [16]. *Baleanu* et al., have studied the numerical study for *fractional Euler-Lagrange equations* of a *Harmonic Oscillator* on moving platform [17].

5. Materials and Methods

The *calculus of variations* is a theory of finding maxima or minima of the given integral. Consider the system that described by an integral function of time, coordinates and derivative. The integral is extremum when the integral satisfies the *Euler-Lagrange equation* [10-13].

Minimized

$$ J(x,t) = \int_{t_1}^{t_2} L(x(t), \dot{x}(t), \ddot{x}(t), t)\,dt \tag{6} $$

Likewise, there exist systems that are feasible to describe their behavior with *fractional differential equations*. Assume that set of given functions defined in the given interval [a, b] which have continuous left and a right *fractional derivative* of order α and β respectively [14].

Theorem 1. Let $J(x,t)$ is integral function of the form

Minimized

$$ J(x,t) = \int_{t_1}^{t_2} L\left(x(t), {}_{a}\mathbf{D}_{t}^{\alpha} x(t), {}_{t}\mathbf{D}_{b}^{\beta} x(t), t\right) dt \tag{7} $$

The necessary condition for integral to have an extremum for a given function, then the integral must satisfy the generalized form of *Euler-Lagrange equation*.

Proof of Theorem 1. By varying the integral with infinitesimal quantity.

$$\delta J(x,t) = \delta \int_{t_1}^{t_2} \mathbb{L}(x(t), {}_a\mathbf{D}_t^{\alpha} x(t), {}_t\mathbf{D}_b^{\beta} x(t), t)dt \tag{8}$$

$$\delta J = \int_{t_1}^{t_2} \left(\frac{\partial \mathbb{L}}{\partial x} \delta x(t) + \frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) + \frac{\partial \mathbb{L}}{\partial\, {}_t\mathbf{D}_b^{\beta} x(t)} \delta\, {}_t\mathbf{D}_b^{\beta} x(t) \right) dt \tag{9}$$

Consider the second term from right hand side

$$\int_{t_1}^{t_2} \frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) dt \tag{10}$$

integrated by part, then we obtained: -

$$u = \frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \qquad du = \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \right) dt \tag{11}$$

$$v = \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) \qquad dv = d(\delta\, {}_a\mathbf{D}_t^{\alpha} x(t)) \tag{12}$$

Thus term in (10) become: -

$$\int_{t_1}^{t_2} \frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) dt = \left[\frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \right) dt \tag{13}$$

From boundary condition, the second term in equation (13) is zero, so we have the following values:

$$\int_{t_1}^{t_2} \frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) dt = -\int_{t_1}^{t_2} \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \right) dt \tag{14}$$

Likewise, the third term of equation (9) become

$$\int_{t_1}^{t_2} \frac{\partial \mathbb{L}}{\partial\, {}_t\mathbf{D}_b^{\beta} x(t)} \delta\, {}_t\mathbf{D}_b^{\beta} x(t) dt = -\int_{t_1}^{t_2} \delta\, {}_t\mathbf{D}_b^{\beta} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_t\mathbf{D}_b^{\beta} x(t)} \right) dt \tag{15}$$

then substitute equation (14) and (15) into equation (9) then we have

$$\delta J = \int_{t_1}^{t_2} \left(\frac{\partial \mathbb{L}}{\partial x} \delta x(t) - \delta\, {}_a\mathbf{D}_t^{\alpha} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \right) - \delta\, {}_t\mathbf{D}_b^{\beta} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_t\mathbf{D}_b^{\beta} x(t)} \right) \right) dt \tag{16}$$

$$\delta J = \int_{t_1}^{t_2} \left(\frac{\partial \mathbb{L}}{\partial x} x(t) + {}_t\mathbf{D}_b^{\alpha} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x(t)} \right) + {}_a\mathbf{D}_t^{\beta} x(t) \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial\, {}_t\mathbf{D}_b^{\beta} x(t)} \right) \right) \delta dt \tag{17}$$

Since $\delta J = 0$, therefore, we obtained the *fractional Euler-Lagrange equation*

$$\frac{\partial \mathbb{L}}{\partial x} + {}_t\mathbf{D}_b^{\alpha} \frac{\partial \mathbb{L}}{\partial\, {}_a\mathbf{D}_t^{\alpha} x} + {}_a\mathbf{D}_t^{\beta} \frac{\partial \mathbb{L}}{\partial\, {}_t\mathbf{D}_b^{\beta} x} = 0 \tag{18}$$

Conclusion

In this article, *fractional calculus* has been applied to find the generalized form of *Euler-Lagrange equation*. By assuming that system is described by *fractional differential* operators. The solution that obtained by solving the optimization problem in *calculus of variations*. The result can be applied to solve engineering optimization problems.

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