QUANTUM CURRENT ALGREBRA SYMMETRY AND DESCRIPTION OF BOLTZMANN TYPE KINETIC EQUATIONS IN STATISTICAL PHYSICS

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We study a special class of dynamical systems of Boltzmann-Bogolubov and Boltzmann-Vlasov type on infinite dimensional functional manifolds modeling kinetic processes in many-particle media. Based on algebraic properties of the canonical quantum symmetry current algebra and its functional representations we proposed a new approach to invariant reducing the Bogolubov hierarchy on a suitably chosen correlation function constraint and deducing the related modified Boltzmann-Bogolubov kinetic equations on a finite set of multiparticle distribution functions.

It is well known that the classical Bogolubov-Boltzmann kinetic equations under the condition of manyparticle correlations [2, 4, 8, 9, 6, 14, 15, 12] at weak short range interaction potentials describe long waves in a dense gas medium. The same equation, called the Vlasov one, as it was shown by N. Bogolubov [9], describes also exact microscopic solutions of the infinite Bogolubov chain [8] for the manyparticle distribution functions, which was widely studied making use of both classical approaches in [4, 6, 7] and in [10, 11, 16, 18], making use of the generating Bogolubov functional method and the related quantum current algebra representations.

A.A. Vlasov proposed his kinetic equation [21] for electron-ion plasma, based on general physical reasonings, that in contrast to the short range interaction forces between neutral gas atoms, interaction forces between charged particles slowly decrease with distance, and therefore the motion of each such particle is determined not only by its pair-wise interaction with either particle, yet also by the interaction with the whole ensemble of charged particles. In this case the Bogolubov equation for distribution functions in a domain $\Lambda \subset \mathbb{R}^3$

$$\frac{\partial f_1(z;t)}{\partial t} + \langle \frac{p}{m} | \nabla_x f_1(z;t) \rangle = \int_{T^*(\Lambda)} dz' \{f_2(z,z';t), V(x-x')\}^{(2)}, \tag{0.1}$$

where $z := (x,p) \in T^*(\Lambda), t \in \mathbb{R}_+$ is the temporal evolution parameter, $\{\cdot,\cdot\}^{(m)}$ denotes the canonical Poisson bracket [1, 3, 5, 6, 20] on the product $T^*(\Lambda)^m, m \in \mathbb{N}$, and $V(x-x'), x, x' \in \Lambda$, is an interparticle interaction potential, - reduces to the Vlasov equation if to put in (0.1)

$$f_2(z,z';t) = f_1(z;t) f_1(z';t), \tag{0.2a}$$

that is to assume that the two-particle correlation function [4] vanishes:

$$g_2(z,z';t) = f_2(z,z';t) - f_1(z;t) f_1(z';t) = 0 \tag{0.3a}$$

for all $z, z' \in T^*(\Lambda)$ and $t \in \mathbb{R}_+$. Then one easily obtains from (0.1) that

$$\frac{\partial f_1(z;t)}{\partial t} + \langle \frac{p}{m} | \nabla_x f_1(z;t) \rangle = \langle \frac{\partial f_1(z;t)}{\partial p} | \nabla_x \int_{T^*(\Lambda)} dz' \ V(x-x') f_1(z';t) \rangle \tag{0.4}$$

for all $z \in T^*(\Lambda)$ and $t \in \mathbb{R}_+$. Remark here that the equation (0.4) is reversible under the time reflection $\mathbb{R}_- \ni -t \rightleftarrows t \in \mathbb{R}_+$, thus it is obvious that it can not describe thermodynamically stable limiting states of the particle system in contrast to the classical Bogolubov-Boltzmann kinetic equations [2, 4, 8, 6, 10, 18], being *a priori* time nonreversible owing to the choice of boundary conditions in the correlation weakening form. This means that in spite of the Hamiltonicity of the Bogolubov chain for the distribution functions, the Bogolubov-Boltzmann equation *a priori* is not reversible. It is also evident that the condition (0.3a) does not break the Hamiltonicity - the equation (0.4) is Hamiltonian with respect to the following Lie-Poisson-Vlasov bracket:

$$\{\{a(f), b(f)\}\} := \int_{T^*(\Lambda)} dz\ f(z)\{\operatorname{grad} a(f)(z), \operatorname{grad} b(f)(z)\}^{(1)}, \tag{0.5}$$

where $\operatorname{grad}(\cdot) := \delta(\cdot)/\delta f, f \in D(T^*(\Lambda)) := M_{f_1}$, respectively $a, b \in D(M_{f_1})$ are smooth functionals on the functional manifold M_{f_1}, consisting of functions fast decreasing at the boundary $\partial\Lambda$ of the domain $\Lambda \subset \mathbb{R}^3$.The bracket expression (0.5) allows a slightly different Lie-algebraic interpretation, based on considering the functional space $D(M_{f_1})$ as a Poissonian manifold, related with the canonical symplectic structure on the diffeomorphism group $Diff(\Lambda)$ of the domain $\Lambda \subset \mathbb{R}^3$, first described [22, 23] still in 1887 by Sophus Lie. Namely, the following classical theorem [1, 17, 23] holds.

Theorem 0.1. *The Lie-Poisson bracket at point $(\mu;\eta) \in T^*_\eta(Diff(\Lambda))$ on the coadjoint space $T^*_\eta(Diff(\Lambda)), \eta \in Diff(\Lambda)$, is equal to the expression*

$$\{f, g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c \tag{0.6}$$

for any smooth right-invariant functionals $f, g \in C^\infty(T^*_\eta(Diff(\Lambda)); \mathbb{R})$.

These aspects and its different consequences are analyzed in detail in our report.

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