IOCA 2021

Research Article / Proceedings

Maximum Multi-Commodity Flow with Proportional and Flow-dependent Capacity Sharing

Durga Prasad Khanal[1], Urmila Pyakurel[2,*], Tanka Nath Dhamala[3], Stephan Dempe[4]

[1] Saraswati Multiple Campus, Tribhuvan University, Kathmandu, Nepal, durgapsdkhanal@gmail.com
[2] Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal, urmilapyakurel@gmail.com
[3] Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal, tanka.nath.dhamala@gmail.com
[4] Faculty of Mathematics and Computer Science, TU Bergakademie Freiberg, Freiberg, Germany, dempe@math.tu-freiberg.de

* Correspondence: urmilapyakurel@gmail.com

Abstract: Multi-commodity flow problems concern with the transshipment of more than one commodities from respective sources to the corresponding sinks without violating the capacity constraints on the arcs. If the objective of the problem is to send the maximum amount of flow within given time horizon, then it becomes the maximum flow problem. In multi-commodity flow problem, flow of different commodities departing from their sources arrive at the common intermediate node and have to share the capacity through the arc. The sharing of the capacity in the common arc (bundle arc) is one of the major issues in the multi-commodity flow problems. In this paper, we introduce the maximum static and maximum dynamic multi-commodity flow problems with proportional capacity sharing and present polynomial time algorithms to solve the problems. Similarly, we investigate the maximum dynamic multi-commodity flow problem with flow-dependent capacity sharing and present a pseudo-polynomial time solution strategy.

Keywords: Multi-commodity, maximum flow, proportional capacity sharing, flow-dependent capacity sharing.

Citation: Lastname, F.; Lastname, F.; Lastname, F. Title. *Proceedings* **2021**, *68*, x. https://doi.org/10.3390/xxxxx

Published: date

1. Introduction

A topological structure with links and crossings, known as arcs and nodes, respectively, is a network in which entities are transshipped from one point to another. The initial and the final points are termed as source and sink nodes, respectively. In a multi-terminal network, transshipment of more than one commodities from respective sources to the corresponding sinks satisfying the capacity constraints on the arcs is a multi-commodity flow (MCF) problem. Supply chain networks, massage routine in telecommunication, transportation networks are some examples of multi-commodity network topology.

Ford and Fulkerson [1] introduced the concept of static multi-commodity flow problem and thereafter many researchers have contributed on the different aspects of the multi-commodity flow problems, [2-5]. If the demand and supply of each commodity is to be maximized in the given time horizon, then the problem becomes a maximum dynamic multi-commodity flow problem. The static multi-commodity flow problem is polynomial time solvable by using the ellipsoid or interior point method, whereas the dynamic multi-commodity flow problem is NP-hard, [6]. Kappmeier [7] provided the solution of maximum dynamic multi-commodity flow problem using time-expanded network in pseudo-polynomial time complexity. Pyakurel et al. [8] presented a polynomial time algorithm for the maximum static flow problem and pseudo-polynomial algorithms for the earliest arrival transshipment and maximum dynamic flow problems with partial contraflow. A priority based multi-commodity flow problem can be found in Khanal et al. [9]. Using the concept of intermediate storage introduced by Pyakurel and Dempe [10], Khanal et al. [11] presented a polynomial time algorithm for maximum static and a pseudo-polynomial time algorithm for maximum dynamic multi-commodity flow problems with intermediate storage.

Sharing of the capacity of bundle arc is one of the major issues in the multi-commodity flow problems. For each commodity, if the sharing of the capacity of bundle arc is set in proportion to the minimum path capacity from their respective sources to the tail node of the bundle arc, then it is known as proportional capacity sharing. In this case, the shared capacity of the bundle arc for each commodity is fixed and the multi-commodity flow problem is reduced to independent single commodity flow problems. To avoid the fractional flow, we can use ceiling and floor functions with appropriate manner. Similarly, if the sharing of the capacity of bundle arc is made according to the inflow rate of the flow of each commodity, then it is termed as flow-dependent capacity sharing. In this method, the shared capacity of the bundle arc may not always be same as the flow on arc may vary over the time. We investigate these two sharing techniques hereafter in Subsections 2.1 and 2.2.

In this paper, we introduce the maximum multi-commodity flow problem using proportional as well as flow-dependent capacity sharing on the bundle arcs. We present the polynomial time algorithms for static as well as dynamic multi-commodity flow problems using proportional capacity sharing in Section 3. Similarly, in Section 4, a pseudo-polynomial time algorithm for the dynamic multi-commodity flow problem with flow-dependent capacity sharing is presented. The paper is concluded in Section 5.

2. Basic Terminologies

Consider a network topology $G = (N, A, K, u, \tau, d_i, S, D, \mathrm{T})$ with N as node set and A as arc set. Here, d_i represents the demand/supply of each commodity $i \in K = \{1, 2, \ldots, \mathrm{k}\}$ which is routed through unique source-sink pair s_i-t_i, where $s_i \in S \subseteq N$ and $t_i \in D \subseteq N$. Each arc $e = (v, w) \in A$ with $head(e) = w$ and $tail(e) = v$ is equipped with a capacity function $u : A \to \mathcal{R}^+$ that restricts the flow of commodity and a non-negative transit time function $\tau : A \to \mathcal{R}^+$ that measures the time to transship the flow from node v to node w. Let $\overrightarrow{\boldsymbol{\delta}}(v)$ and $\overleftarrow{\boldsymbol{\delta}}(v)$ be the set of outgoing arcs from node v and incoming arcs to node v, respectively. We denote $\boldsymbol{P}_i$ as the set of all paths of commodity i such that $\mathrm{P} \in \boldsymbol{P}_i$ is a s_i-t_i, path and $\mathrm{P}_{[si,v]} \in \boldsymbol{P}_i$ represents the path from s_i to the intermediate node v. The time horizon is denoted by $\boldsymbol{T} = \{0, 1, \ldots, \mathrm{T}\}$ in discrete time settings and $\boldsymbol{T} = [0, \mathrm{T}+1)$ in continuous time settings. In case of static flow, time parameters T and $\boldsymbol{\tau}$ are absent.

2.1. Proportional Capacity Sharing

Multi-commodity flow problem differs from single commodity flow problem due to the bundle constraints and the unique source-sink flow for each commodity. Our assumption is that the nature of flows inside the same commodity group are homogeneous and between the commodity groups are heterogeneous but uniform in the occupancy rate of arc capacity. To share the capacity of bundle arc, we propose a proportional capacity sharing technique depending on the minimum capacity of paths $\mathrm{P}_{[si,v]}$, for each commodity i, from their respective sources s_i to the tail v of bundle arc $e = (v, w)$ as follows. Let u_e be the capacity of a bundle arc e, then proportional sharing of capacity u_e for each commodity $i \in K$ is,

$$u_e^i = \frac{u_a^i}{\sum_{a \in \mathrm{P}_{[si,v]}: i \in K} \mathrm{u}_a^i} u_e \qquad (1)$$

where, $\mathrm{P}_{[si,v]}$ is the path from s_i to the tail v of bundle arc e, for all $i \in K$ and a is an arc in $\mathrm{P}_{[si,v]}$ with minimum capacity. Here, u_e^i represents the portion of capacity of arc e allocated for the commodity i. Clearly, the sum of shared capacities over each commodity is equal to the original arc capacity, i.e., $\sum_{i \in K} u_e^i = u_e$.

The shared capacity may be in fraction, i.e., $u_e^i = int(u_e^i) + fra(u_e^i)$, the sum of integral part and fractional part, respectively. The fractional capacities can be converted into integral capacities as follows.

- Find the sum $\sum fra(u_e^i)$. If $\sum fra(u_e^i)$ = p, then first p fractional capacities with greatest fractional part (with descending order of fractional part) are rounded up using ceiling function $\lceil . \rceil$ and remaining capacities are rounded below by floor function $\lfloor . \rfloor$.

- If same fractional part occurs in more than one commodities then priority is given to capacity with greatest integral part among them.

- In case of equal integral parts, priority goes to the commodity with higher demand among them. If the demand values are also same then either of them can be rounded up.

It is to be noted that, if $u_e^i < 1$ and has no alternative path for commodity *i*, then it may block the transshipment of the flow. In such a case, the fractional capacity is to be accepted.

2.2. Flow-dependent Capacity Sharing

In proportional capacity sharing technique, the shared capacity of each commodity remains fixed at each time step θ. In this subsection, we present the flow-dependent capacity sharing technique where the share of capacity for each commodity depends on the inflow rate of the flow *f* in predecessor arcs. At any instance of time θ, if a bundle arc $e = (v, w)$ with capacity u_e holds more than one commodities $i \in K$, then the flow-dependent capacity sharing of u_e for each commodity $i \in K$ is,

$$u_e^i(\theta) = \frac{f_a^i(\theta-\tau_a)}{\sum_{a\epsilon\, \alpha(e):i \in K} f_a^i(\theta-\tau_a)} u_e \qquad (2)$$

where, $\alpha(e)$ is the set of predecessor arcs of bundle arc *e* so that $a\epsilon\, \alpha(e) \Rightarrow head(a) = tail(e)$ and $u_e^i(\theta)$ is the portion of capacity of arc e for commodity *i* at time θ. For each time θ, the sum of the portion of shared capacities $u_e^i(\theta)$ over all commodities $i \in K$ is equal to the original arc capacity, i.e., $\sum_{i \in K} u_e^i(\theta) = u_e$. If the shared capacities are in fraction, we can convert them into integer values as described in Subsection 2.1.

3. Maximum MCF with Proportional Capacity Sharing

3.1. Maximum Static Multi-commodity Flow

In static network $G = (N, A, K, u, d_i, S, D)$, the multi-commodity flow φ with proportional capacity sharing is the sum of non-negative flows $\varphi^i : A \rightarrow \mathcal{R}^+$ for each i with demand d_i satisfying the proportional capacity sharing Equation (1) together with the conditions (3 - 4).

$$\sum_{e\in\vec{\delta}(v)} \varphi_e^i - \sum_{e\in\overleftarrow{\delta}(v)} \varphi_e^i = \begin{cases} d_i & for\ \ v = s_i \\ -d_i & for\ \ v = t_i \\ 0 & otherwise \end{cases} \qquad \forall\, v \in N, i \in K \qquad (3)$$

$$0 \le \varphi_e^i \le u_e^i \ \ \forall\, e \in A, i \in K \qquad (4)$$

Constraints in (3) represent the supply/demand at source/sink nodes and the flow conservation constraints at intermediate nodes whereas the constraints in (4) represent the boundedness of the flow on arcs by their capacities. By taking sum over each commodity in later equation, we get the bundle constraints $0 \le \sum_{i\in K} \varphi_e^i \le \sum_{i\in K} u_e^i = u_e$ for all $e \in A$. For a maximum static multi-commodity flow problem with proportional capacity sharing, the objective is to maximize the total flow value $\sum_{i\in K} d_i = |\varphi|$ subject to the constraints (1, 3 and 4).

We now introduce the maximum static multi-commodity flow problem with proportional capacity sharing as follows.

Problem 1: For given static multi-commodity network $G = (N, A, K, u, d_i, S, D)$, the maximum static multi-commodity flow problem with proportional capacity sharing is to tranship the maximum flow from s_i to t_i, where shared capacity for each commodity $i \in K$ on the bundle arc is depending on the minimum capacity of paths from respective source to the tail node of the bundle arc.

To solve the problem, we first reduce the multi-commodity flow problem into k independent single commodity flow problems by sharing the capacity of the bundle arc using Equation (1). For each commodity i, maximum static flow φ^i is obtained and the sum of the flows for the commodities is the maximum static flow value $|\varphi|$. We now present the algorithm to solve Problem 1.

Algorithm 1: Maximum static MCF algorithm with proportional capacity sharing

Input: Given static multi-commodity flow network $G = (N, A, K, u, d_i, S, D)$.

1. Construct k independent sub-problems by proportional capacity sharing (1) on bundle arcs for all $i \in K$.
2. Compute the solution φ^i to the static maximum flow problem for all i.
3. Maximum flow $|\varphi| = \sum_{i \in K} \varphi^i$.

Output: Maximum static MCF on G with proportional capacity sharing.

Theorem 1. Algorithm 1 solves the maximum static MCF problem correctly in polynomial time complexity.

3.2. Maximum Dynamic Multi-commodity Flow

For a given dynamic network G with constant transit times τ on arc e, the MCF over time function f with proportional capacity sharing is the sum of flows $f^i: A \times \boldsymbol{T} \to \mathcal{R}^+$, satisfying the proportional capacity sharing Equation (1) together with the constraints (5 - 7).

$$\sum_{e \in \vec{\delta}(v)} \sum_{\theta=0}^{T} f_e^i(\theta) - \sum_{e \in \overleftarrow{\delta}(v)} \sum_{\theta=0}^{T} f_e^i(\theta) = \begin{cases} d_i & for \ v = s_i \\ -d_i & for \ v = t_i \\ 0 & otherwise \end{cases} \qquad \forall\, v \in N, i \in K \tag{5}$$

$$\sum_{e \in \vec{\delta}(v)} \sum_{\theta=0}^{\beta} f_e^i(\theta) - \sum_{e \in \overleftarrow{\delta}(v)} \sum_{\theta=0}^{\beta} f_e^i(\theta) \le 0 \qquad \forall\, v \notin \{s_i, t_i\}, i \in K, \beta \in \boldsymbol{T} \tag{6}$$

$$0 \le f_e^i(\theta) \le u_e^i \qquad \forall\, e \in A, i \in K \ and \ \theta \in \boldsymbol{T} \tag{7}$$

Here, the constraints in (5) represent the supply/demand at sources/sinks and the flow conservation at intermediate nodes on time horizon T. The non-conservation of the flow at intermediate nodes in any time step β in $\boldsymbol{T} = \{0, 1, \ldots, T\}$ are represented by the constraints in (6). Similarly, (7) represent that the flows on arcs are bounded above by their capacities. With these constraints together with Equation (1), we introduce the maximum dynamic MCF problem with proportional capacity sharing which maximizes the total flow value $\sum_{i \in K} d_i = |f|$ within given time horizon T as follows.

Problem 2: For given dynamic multi-commodity network G = (N, A, K, u, τ, d_i, S, D, T*), the maximum multi-commodity flow problem with proportional capacity sharing is to transship the maximum amount of flow from s_i to t_i within given time horizon* T*, where shared capacity for each $i \in K$ on the bundle arc is depending on the minimum capacity of paths from respective source to the tail node of the bundle arc.*

We now present an algorithm to solve Problem (2).

Algorithm 2: Maximum dynamic MCF algorithm with proportional capacity sharing

Input: Given dynamic multi-commodity flow network $G = (N, A, K, u, \tau, d_i, S, D, T)$.

1. Construct k independent sub-problems by proportional capacity sharing (1) on bundle arcs for all $i \in K$.
2. Compute the maximum static flow φ^i for all i using Algorithm (1).
3. Decompose flow φ^i into path flows φ_P^i.
4. Determine maximum dynamic flow for each $i \in K$ using temporally repeated flow such that

   $f^i = \sum_{P \in \boldsymbol{P}_i} (T + 1 - \tau_P)\varphi_P^i$.
5. Maximum flow $|f| = \sum_{i\in K} f^i$.

Output: Maximum dynamic MCF on G with proportional capacity sharing.

Theorem 2: Algorithm 2 provides the feasible solution to the maximum dynamic MCF problem with proportional capacity sharing in polynomial time.

4. Maximum MCF with Flow-dependent Capacity Sharing

For a given dynamic network G with constant transit times τ on arc e, the multi-commodity flow over time function f with flow-dependent capacity sharing is the sum of flows $f^i: A \times \boldsymbol{T} \to \mathcal{R}^+$, satisfying the constraints (8 - 12).

$$\sum_{e\in\vec{\delta}(v)}\sum_{\theta=0}^{T} f_e^i(\theta) - \sum_{e\in\overleftarrow{\delta}(v)}\sum_{\theta=0}^{T} f_e^i(\theta) = \begin{cases} d_i & for\ v = s_i \\ -d_i & for\ v = t_i \\ 0 & otherwise \end{cases} \qquad \forall\, v \in N, i \in K \tag{8}$$

$$\sum_{e\in\vec{\delta}(v)}\sum_{\theta=0}^{\beta} f_e^i(\theta) - \sum_{e\in\overleftarrow{\delta}(v)}\sum_{\theta=0}^{\beta} f_e^i(\theta) \le 0 \qquad \forall\, v \notin \{s_i, t_i\}, i \in K, \beta \in \boldsymbol{T} \tag{9}$$

$$\sum_{i\in K} f_e^i(\theta) \le u_e \qquad \forall e \in A \tag{10}$$

$$u_e^i(\theta) = \frac{f_a^i(\theta - \tau_a)}{\sum_{a\epsilon\, \alpha(e):i\,\in K} f_a^i(\theta - \tau_a)} u_e \qquad \forall e \in A \tag{11}$$

$$f_e^i(\theta) \ge 0 \qquad \forall\, e \in A,\ \ i \in K\ and\ \ \theta \in \boldsymbol{T} \tag{12}$$

Here, the constraints in (8) and (9) have usual meanings as represent in Subsection (3.2). The bundle constraints bounded by arc capacities are presented by (10). Constraints in (11) represent the flow-dependent capacity sharing and non-negativity of flows are represented by the constraints in (12). We now present the maximum dynamic MCF problem with flow-dependent capacity sharing satisfying the above constraints as follows.

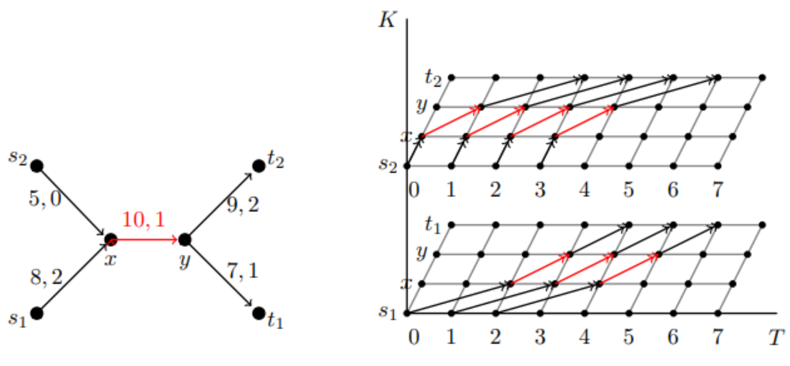
Problem 3. For a given multi-commodity network G = (N, A, K, u, τ, d_i, S, D, T), the maximum multi-commodity flow problem with flow-dependent capacity sharing is to transship the maximum amount of flow from s_i to t_i within given time horizon T, where shared capacity for each $i \in K$ on the bundle arc is depending on the inflow of incoming arcs of the bundle arc.

To solve the problem, we use time expanded layer graph.

4.1 *Multi-commodity Time-expanded Layer Graph*

The multi-commodity time-expanded layer graph is a three dimensional graph that contains the copy of nodes from underlying static network for every discrete time steps and for each commodity. It is applicable to solve the variety of flow over time problems by applying the algorithms and techniques developed for the static network flows. For a given network G with integral transit time on arcs and time horizon T, the T-time expanded layer graph G^T is obtained by creating T+1 copies of node set N which are labeled as $N(0), N(1), \dots, N(T)$ together with θ^{th} copy of node v labeled as $v(\theta)$, $\theta \in \boldsymbol{T}$ and the commodities $i \in K$. For every arc $e = (v, w) \in A$ and $\theta \in \{0, 1, \dots, T - \tau_e\}$, there is an arc $e_i(\theta)$ from $v_i(\theta)$ to $w_i(\theta + \tau_e)$ with the same capacity of arc e for single commodity arc and the sharing capacity for bundle arc e. If intermediate storage is allowed at node v, then the arc from $v_i(\theta)$ to $v_i(\theta + 1)$ represents the holdover arc with infinite capacity that is used to hold the flow for unit time interval $[\theta, \theta + 1)$ for all $\theta \in \{0, 1, \dots, \mathrm{T}\}$.

For the graphical representation, we present a three dimensional layer graph G^T with the set of node N, time T and commodity K as the coordinate axes (see Figure 1). Each commodity $i \in K$ preforms the layers of graphs in vertical line. In Figure 1, network (a) represents a two-commodity network in which commodity-1 is transshipped from s_1 to t_1 and commodity-2 from s_2 to t_2. Arc (x, y) is the bundle arc which carries both commodities. Figure 1(b) represents the time-expanded layer graph of Figure 1(a) with time horizon T = 6, where parallel arcs on (x, y) share the capacity for each commodity with flow-dependent capacity sharing technique. At time step $\theta = 0$ and $\theta = 1$, no flow of commodity-1 reaches to arc (x, y) so only commodity-2 is transshipped on it. But then after, capacity is shared among the commodities. Similarly, bundle arc transships only commodity-1 at time $\theta = 4$ due to the absence of commodity-2.



(a) capacity, transit time (b) layer graph

Figure 1: (b) represents the time-expanded layer graph G^T of given network (a).

Depending on the time-expanded layer graph, we now present the algorithm to solve Problem 3.

Algorithm 3: Maximum dynamic MCF algorithm with flow-dependent capacity sharing

Input: Given dynamic multi-commodity flow network $G = (N, A, K, u, \tau, d_i, S, D, \mathrm{T})$.

1. Construct a multi-commodity time-expanded layer graph G^T.
2. Share the capacity on bundle arcs (parallel arcs in G^T) with flow-dependent capacity sharing (2) at each $\theta \in \boldsymbol{T}$.
3. Decompose the static flow φ^i into path flows $\varphi^i_P(\theta)$ in G^T at each time step θ.
4. Maximum flow $|f| = \sum_{i \in K} \varphi^i_P$.

Output: Maximum dynamic MCF on G with proportional capacity sharing.

Theorem 3. A feasible solution to the maximum dynamic MCF problem with flow-dependent capacity sharing can be obtained by using Algorithm 3 in pseudo-polynomial time.

5. Conclusion

Maximum MCF problem deals with the transshipment of maximum amount of flow of more than one different commodities from respective sources to the corresponding sinks within given time horizon. Allocation of the capacity of bundle arc to each commodity is one of the major issues in multi-commodity flow problem. To deal with this problem, we have proposed proportional capacity sharing and flow-dependent capacity sharing. We have presented polynomial time solutions for static as well as dynamic maximum MCF problems with proportional capacity sharing and a pseudo-polynomial time algorithm with flow-dependent capacity sharing. To the best of our knowledge, these solution strategies for the maximum MCF problems are introduced for the first time.

Acknowledgements

The second author (Urmila Pyakurel) thanks to the Alexander von Humboldt Foundation for Digital Cooperation Fellowship (August 1, 2021 – January 31, 2022).

References

1. Ford, L.R.; Fulkerson, D.R. *Flows in Networks*, Princeton University Press, Princeton, New jersey, 1962.
2. Ahuja, R.K.; Magnanti, T.L.; Orlin, J.B. *Network Flows: Theory, Algorithm and Applications*, Prentice Hall, Englewood Cliffs, 1993.
3. Ali, A.; Helgason, R.; Kennington, J.; Lall, H. Computational comparison among three multi-commodity network flow algorithms. *Operations Research*, **1980**, *28(4)*, 995-1000.
4. Assad, A. Multi-commodity network flows- a survey. *Networks*, **1978**, *8(1)*, 37-91.
5. Kennington, J. A survey of linear cost multi-commodity network flows. *Operations Research*, **1978**, *26(2)*, 209-236.
6. Hall, A.; Hippler, S.; Skutella, M. Multi-commodity flows over time: Efficient algorithms and complexity. *Theoretical Computer Science*, **2007**, *379*, 387-404.
7. Kappmeier, P.W. Generalizations of flows over time with application in evacuation optimization. PhD Thesis, Technical University, Berlin, Germany, 2015.
8. Pyakurel, U; Gupta, S.P.; Khanal, D.P.; Dhamala, T.N. Efficient algorithms on multicommodity flow over time problems with partial lane reversals. *International Journal of Mathematics and Mathematical Sciences, Hindawi*, **2020**, DOI: https://doi.org/10.1155/2020/2676378.
9. Khanal, D.P.; Pyakurel, U.; Dhamala, T.N. Prioritized multi-commodity flow model and algorithm. Presented on: International Symposium on Analytic Hierarchy Process 2020 (ISAHP 2020), DOI: https://doi.org/10.13033/isahp.y2020.049.
10. Pyakurel, U.; Dempe, S. Network flow with intermediate storage: models and algorithms. *SN Operations Research Forum*, **2020**, DOI: https://doi.org/10.1007/s43069-020-00033-0.
11. Khanal, D.P.; Pyakurel, U.; Dhamala, T.N. Maximum multicommodity flow with intermediate storage. *Mathematical Problems in Engineering, Hindawi*, **2021**, DOI: https://doi.org/10.1155/2021/5063207.