Maximum Multi-Commodity Flow with Proportional and Flow-dependent Capacity Sharing

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Outlines

1. Motivation
2. Literature Review
3. Proportional Capacity Sharing:
   - Problem Statement
   - Flow Model
   - Solution Strategy
4. Flow-dependent Capacity Sharing:
   - Problem Statement
   - Flow Model
   - Solution Strategy

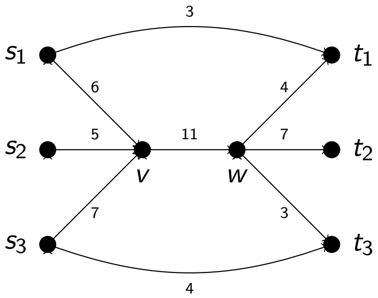
Motivation

- Transshipment of more than one commodities from respective sources to the corresponding sinks is a multi-commodity flow problem.
- Maximum multi-commodity flow problem is applicable for the emergency management, supply chain of goods and transportation network.
- Sharing of the bundle arc capacity for each commodity is one of the most important issue in multi-commodity flow problem.
- Proportional capacity sharing technique shares the capacity of bundle arc in proportion to the minimum capacity of each path from respective source to the bundle arc and reduce the problem in to k independent sub-problems.
- Flow-dependent capacity sharing technique shares the capacity of bundle arc with respect to the inflow rate of the flow in predecessor arcs.

Literature Review

- Ford and Fulkerson [1962]: Maxflow mincut, quickest flow, time expanded network, temporally repeated flow, multi-commodity flow.
- Fleischer and Skutella [2002]: Multi-commodity flow over time problem.
- Hall et al. [2007]: Hardness of multi-commodity flow over time problem.
- Fleischer and Skutella [2007]: Approximation solution to QMCF problem.
- Kappmeier [2015]: Solution of maximum dynamic multi-commodity flow problem using time-expanded network in pseudo-polynomial time complexity.
- Dhamala et al. [2020]: Polynomial time algorithm for quickest multi-commodity contraflow problem.
- Pyakurel and Dempe [2020]: Flow with intermediate storage.
- Khanal et al. [2021]: Maximum multi-commodity flow with intermediate storage.

Multi-commodity Network



$\mathcal{N} = (V, A, K, u, d_i, S, D)$
where,
$\mathcal{N}$ = Network
V = Set of nodes
A = Set of arcs
u = Capacity of arc
S = Set of source nodes s_i
D = Set of sink nodes t_i
K = Set of commodities
d_i = Net flow of commodity i

I. Proportional Capacity Sharing

Let u_e be the capacity of a bundle arc e, then proportional sharing of capacity u_e for each commodity $i \in K$ is,

$$u_e^i = \frac{u_a^i}{\sum\limits_{a \in P_{[s_i, v]} : i \in K} u_a^i} u_e \tag{1}$$

where, $P_{[s_i,v]}$ is the path from s_i to the tail v of bundle arc e, $\forall i \in K$ and a is an arc in $P_{[s_i,v]}$ with minimum capacity.

Problem Statement

Problem 1

For given static multi-commodity network $\mathcal{G} = (N, A, K, u, d_i, S, D)$, the maximum static multi-commodity flow problem with proportional capacity sharing is to transship the maximum flow from s_i to t_i, where shared capacity for each commodity $i \in K$ on the bundle arc is depending on the minimum capacity of paths from respective source to the tail node of the bundle arc.

Static Flow Model

In static network $\mathcal{G} = (N, A, K, u, d_i, S, D)$, the multi-commodity flow φ with proportional capacity sharing is the sum of non-negative flows $\varphi^i : A \to \mathcal{R}^+$ for each i with demand d_i satisfying the proportional capacity sharing Equation (1) together with the conditions (2 - 3).

$$\sum_{e \in \overrightarrow{\delta}(v)} \varphi_e^i - \sum_{e \in \overleftarrow{\delta}(v)} \varphi_e^i = \begin{cases} d_i & \text{if } v = s_i \\ -d_i & \text{if } v = t_i \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in N, i \in K \tag{2}$$

$$0 \leq \varphi_e^i \leq u_e^i \quad \forall e \in A, i \in K \tag{3}$$

Algorithm 1: Static

Input: Given static multi-commodity flow network $\mathcal{G} = (N, A, K, u, d_i, S, D)$.
Output: Maximum static MCF on $\mathcal{G}$ with proportional capacity sharing.

1. Construct k independent sub-problems by proportional capacity sharing (1) on bundle arcs for all $i \in K$.
2. Compute the solution φ_i to the static maximum flow problem for all i.
3. Maximum flow $|\varphi| = \sum_{i \in K} \varphi_i$.

Theorem: Algorithm 1 solves the maximum static MCF problem correctly in polynomial time complexity.

Dynamic Flow Model

For a given dynamic network $\mathcal{G}$ with constant transit times τ on arc e, the MCF over time function f with proportional capacity sharing is the sum of flows $f^i : A \times \mathbb{T} \to \mathcal{R}^+$, satisfying the proportional capacity sharing Equation (1) together with the constraints (4 - 6).

$$\sum_{e \in \overrightarrow{\delta}(v)} \sum_{\theta=0}^{T} f_e^i(\theta) \quad - \quad \sum_{e \in \overleftarrow{\delta}(v)} \sum_{\theta=0}^{T} f_e^i(\theta) = \begin{cases} d_i & \text{if } v = s_i \\ -d_i & \text{if } v = t_i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in K, \ \forall v \in N \tag{4}$$

$$\sum_{e \in \overrightarrow{\delta}(v)} \sum_{\theta=0}^{\beta} f_e^i(\theta) \quad - \quad \sum_{e \in \overleftarrow{\delta}(v)} \sum_{\theta=0}^{\beta} f_e^i(\theta) \leq 0 \ \forall \ v \notin \{s_i, t_i\}, \ \ i \in K, \ \ \beta \in \mathbb{T} \tag{5}$$

$$0 \leq f_e^i(\theta) \quad \leq \quad u_e^i \ \ \forall e \in A, \ \ \theta \in \mathbb{T} \ \text{ and } \ \ i \in K. \tag{6}$$

Algorithm 2: Dynamic

Input: Given dynamic multi-commodity flow network $\mathcal{G} = (N, A, K, u, \tau, d_i, S, D, T)$.
Output: Maximum dynamic MCF on $\mathcal{G}$ with proportional capacity sharing.

1. Construct k independent sub-problems by proportional capacity sharing (1) on bundle arcs for all $i \in K$.
2. Compute the maximum static flow φ_i for all i using Algorithm 1.
3. Decompose flow φ_i into path flows φ_P^i.
4. Determine maximum dynamic flow for each $i \in K$ using temporally repeated flow such that $f^i = \sum_{P \in \mathbf{P}_i} (T + 1 - \tau_P)\varphi_P^i$.
5. $|f| = \sum_{i \in K} f^i$.

Theorem: Algorithm 2 provides the feasible solution to the maximum dynamic MCF problem with proportional capacity sharing in polynomial time.

II. Flow-dependent Capacity Sharing

At any instance of time θ, if a bundle arc $e = (v, w)$ with capacity u_e holds more than one commodities $i \in K$, then the flow-dependent capacity sharing of u_e for each commodity $i \in K$ is,

$$u_e^i(\theta) = \frac{f_a^i(\theta - \tau_a)}{\sum\limits_{a \in \alpha(e): i \in K} f_a^i(\theta - \tau_a)} u_e \tag{7}$$

where, $\alpha(e)$ is the set of predecessor arcs of bundle arc e so that $a \in \alpha(e) \implies head(a) = tail(e)$ and $u_e^i(\theta)$ is the portion of capacity of arc e for commodity i at time θ.

Problem Statement

Problem 2

For given static multi-commodity network $\mathcal{G} = (N, A, K, u, d_i, S, D)$, the maximum static multi-commodity flow problem with proportional capacity sharing is to transship the maximum flow from s_i to t_i, where shared capacity for each commodity $i \in K$ on the bundle arc is depending on the minimum capacity of paths from respective source to the tail node of the bundle arc.

Flow Model

For a given dynamic network $\mathcal{G}$ with constant transit times τ on arc e, the multi-commodity flow over time function f with flow-dependent capacity sharing is the sum of flows $f^i : A \times \mathbb{T} \to \mathcal{R}^+$, satisfying the constraints (8 - 12).

$$\sum_{e \in \overrightarrow{\delta}(v)} \sum_{\theta=0}^{T} f_e^i(\theta) - \sum_{e \in \overleftarrow{\delta}(v)} \sum_{\theta=0}^{T} f_e^i(\theta) = \begin{cases} d_i & \text{if } v = s_i \\ -d_i & \text{if } v = t_i \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in K, \forall v \in N \tag{8}$$

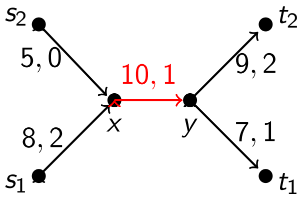
$$\sum_{e \in \overrightarrow{\delta}(v)} \sum_{\theta=0}^{\beta} f_e^i(\theta) - \sum_{e \in \overleftarrow{\delta}(v)} \sum_{\theta=0}^{\beta} f_e^i(\theta) \leq 0 \ \forall \ v \notin \{s_i, t_i\}, \ i \in K, \ \beta \in \mathbb{T} \tag{9}$$

$$\sum_{i \in K} f_e^i(\theta) \leq u_e \ \forall e \in A \tag{10}$$

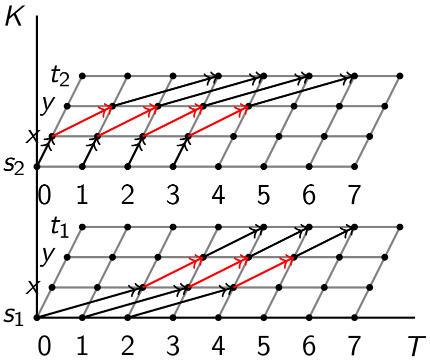
$$u_e^i(\theta) = \frac{f_a^i(\theta - \tau_a)}{\sum_{a \in \alpha(e): i \in K} f_a^i(\theta - \tau_a)} u_e \ \forall e \in A \tag{11}$$

$$f_e^i(\theta) \geq 0 \ \forall i \in K \ \text{ and } \ e \in A. \tag{12}$$

Solution Strategy: Time Expanded Layer Graph



(a) capacity, transit time



(b) layer graph

Figure: (b) represents the time-expanded layer graph $\mathcal{G}^{\mathcal{T}}$ of given network (a).

Algorithm 3

Input: Given dynamic multi-commodity flow network
$\mathcal{G} = (N, A, K, u, \tau, d_i, S, D, T)$.
Output: Maximum dynamic MCF on $\mathcal{G}$ with proportional capacity sharing.

1. Construct a multi-commodity time-expanded layer graph $\mathcal{G}^T$.
2. Share the capacity on bundle arcs (parallel arcs in $\mathcal{G}^T$) with flow-dependent capacity sharing (7) at each $\theta \in \mathbb{T}$.
3. Decompose the static flow φ_i into path flows $\varphi^i_P(\theta)$ in $\mathcal{G}^T$ at each time step θ.
4. $|f| = \sum_{i \in K} \varphi^i_P$.

Theorem:A feasible solution to the maximum dynamic MCF problem with flow-dependent capacity sharing can be obtained by using Algorithm 3 in pseudo-polynomial time.

References I

- L. R. Ford and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, New jersey, 1962.
- R. K. Ahuja, T. L. Magnanti and J. B. Orlin, *Network Flows: Theory, Algorithm and Applications*, Prentice Hall, Englewood Cliffs, 1993.
- A. Ali, R. Helgason, J. Kennington and H. Lall, "Computational comparison among three multi-commodity network flow algorithms", *Operations Research*, vol. 28, no. 4, pp. 995-1000, 1980.
- A. Assad, "Multi-commodity network flows- a survey", *Networks*, vol. 8, no.1, pp. 37-91, 1978.
- J. Kennington, "A survey of linear cost multi-commodity network flows", *Operations Research*, vol. 26, no. 2, pp. 209-236, 1978.
- A. Hall, S. Hippler and M. Skutella, "Multi-commodity flows over time: Efficient algorithms and complexity", *Theoretical Computer Science*, vol. 379, pp. 387-404, 2007.
- P. W. Kappmeier, *Generalizations of flows over time with application in evacuation optimization*, PhD Thesis, Technical University, Berlin, Germany, 2015.

References II

U. Pyakurel, S. P. Gupta, D. P. Khanal and T. N. Dhamala, "Efficient algorithms on multicommodity flow over time problems with partial lane reversals", *International Journal of Mathematics and Mathematical Sciences, Hindawi*, 2020, DOI: https://doi.org/10.1155/2020/2676378.

D. P. Khanal, U. Pyakurel, T. N. Dhamala, "Prioritized multi-commodity flow model and algorithm", *Presented on: International Symposium on Analytic Hierarchy Process 2020 (ISAHP 2020)*, https://doi.org/10.13033/isahp.y2020.049.

U. Pyakurel and S. Dempe, "Network Flow with Intermediate Storage: Models and Algorithms", *SN Operations Research Forum*, 2020, DOI:10.1007/s43069-020-00033-0.

D. P. Khanal, U. Pyakurel, T. N. Dhamala, "Maximum Multicommodity Flow with Intermediate Storage", *Mathematical Problems in Engineering, Hindawi* 2021, DOI: https://doi.org/10.1155/2021/5063207.

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