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Partial differential equations of motion for a single-link flexible manipulator

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- ## Introduction

Lightweight manipulators are the focus of robotics research due to their low usage of energy and positive economic aspect regardless of the complexity of their mechanical model.

The mechanical modeling of any serial link flexible manipulator is based on the one of a single-link flexible manipulator.

The mechanical modeling of a single-link in this work is based on the Euler-Bernoulli beams kinematics.

In the state of art, the kinematics of the Euler-Bernoulli beam is usually approached by the assumed traditional deformation field that cannot allow having an orthogonal elastic rotation matrix to the second-order.

The kinematic model in this work is based on the complete second-order deformation field.

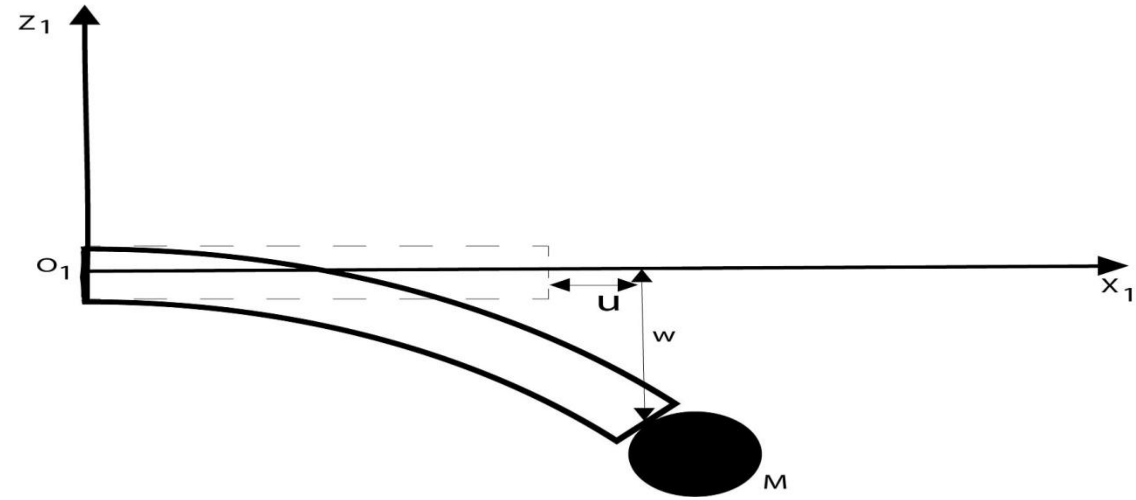
- # Mechanical modeling

The system consists of a base subjected to an applied torque T_{mot} by a motor, a flexible link modeled as an Euler-Bernoulli beam with a circular cross-section with radius R, and length L and a payload with mass m_p and inertia matrix I_p at the free end of the link.

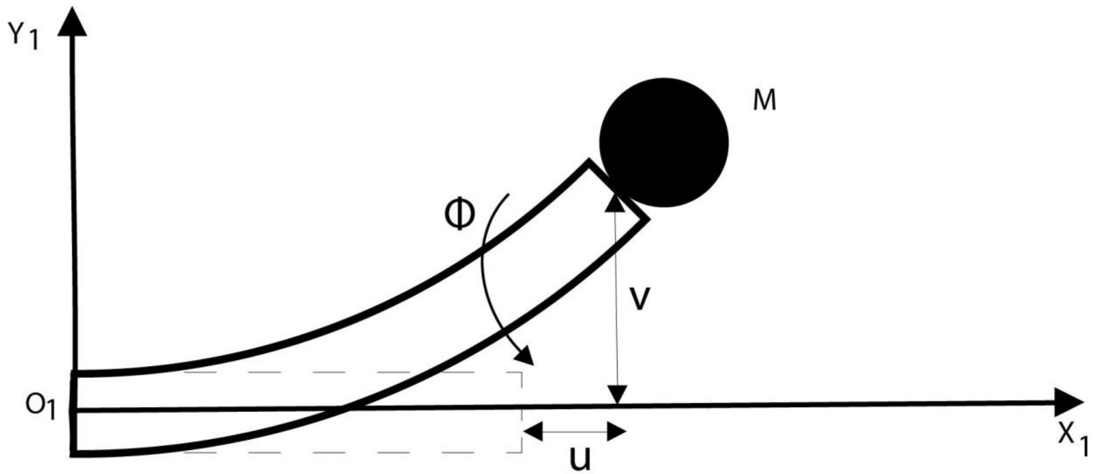
The beam is subjected to an axial stretching $u(x,t)$, a horizontal deflection $v(x,t)$, a vertical deflection $w(x,t)$ and a torsional deformation $\phi(x,t)$.

The beam deformations and their partial derivatives are assumed to be small, shear due to bending, warping due to torsion, air viscous friction are neglected.

1. Kinematics



Front view



Top View

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Let R_0 be an inertial frame with origin O_0, R_1 a frame attached to the motor with origin O_1 that coincides with O_0 and R_{dm} a frame attached to the cross-section of mass dm whose axes are parallel to those of R_1 before deformation and whose origin O_{dm} is the center of the cross-section that is at a distance x from O_1 along the neutral axis of the link before deformation.

The position of O_{dm} relative to R_1 expressed in R_1 after deformation is:

$$^1\overrightarrow{O_1O_{dm}} = \left[x + u - \frac{1}{2}\int_0^x (v'^2 + w'^2)ds, v, w\right]^T$$

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The rotation matrix of R_{dm} relative to R_1 after deformation

$$
{}^1R_{dm} = \begin{bmatrix} 1-\frac{1}{2}(v'^2+w'^2) & -v'+u'v'-w'\phi & -w'+u'w'+v'\phi \\ v'-u'v' & 1-\frac{1}{2}(v'^2+\phi^2) & -\phi-\frac{1}{2}v'w' \\ w'-u'w' & \phi-\frac{1}{2}v'w' & 1-\frac{1}{2}(w'^2+\phi^2) \end{bmatrix}
$$

R_{dm} is verified to be orthogonal to the second-order of Taylor expansion in the deformation variables.

Let P be a point of the cross-section with (x,y,z) its coordinates relative to R_1 before deformation.

The position of P relative to R_1 expressed in R_1 after deformation is given by:

$$
\begin{aligned}
{}^{1}\overrightarrow{O_1P} &= {}^{1}\overrightarrow{O_1O_{dm}} + {}^{1}R_{dm}\,{}^{dm}\overrightarrow{O_{dm}P} \\
&= \begin{bmatrix} x + u - \frac{1}{2}\int_0^x (v'^2 + w'^2)ds + y(-v' + u'v' - w'\phi) + z(-w' + u'w' + v'\phi) \\ v + y(1 - \frac{1}{2}(v'^2 + \phi^2)) + z(-\phi - \frac{1}{2}v'w') \\ w + y(\phi - \frac{1}{2}v'w') + z(1 - \frac{1}{2}(w'^2 + \phi^2)) \end{bmatrix}
\end{aligned}
$$

Let R_2 be a frame attached to the free end of the link whose origin is O_2 and obtained from R_{dm} by replacing x by L.

If the position payload center of mass C relative to R_2 expressed in R_2 is given by:

$$^{2}\overrightarrow{O_2C} = [c,0,0]^T$$

Then the position of C relative to R_1 expressed in R_1 is:

$$^{1}\overrightarrow{O_1C} = [L + u_L - \frac{1}{2}\int_0^L (v'^2 + w'^2)ds + c(1 - \frac{1}{2}(v'^2_L + w'^2_L)), v_L + c(v'_L - u'_L v'_L), w_L + c(w'_L - u'_L w'_L)]^T$$

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The angular velocity of R_1 relative to R_0 expressed in R_0 is:

$$ {}^{0}\overrightarrow{\Omega_{1/0}} = [0, 0, \dot{\theta}]^T $$

The angular velocity of R_{dm} relative to R_1 expressed in R_1 is found using this matrix

$$ S = {}^{1}\dot{R}_{dm} \, {}^{1}R_{dm}^{T} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} $$

Hence

$$ {}^{1}\overrightarrow{\Omega_{dm/1}} = [\omega_x, \omega_y, \omega_z]^T $$

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The Taylor expansion of ${}^{1}\overrightarrow{\Omega_{dm/1}}$ to the second-order in the deformation variables and after simplification gives:

$$\omega_x \approx \dot{\phi} + \frac{1}{2}\left(v'\dot{w}' - \dot{v}'w'\right) \qquad \omega_y \approx -\dot{w}' + \dot{u}'w' + u'\dot{w}' + v'\dot{\phi} \qquad \omega_z \approx \dot{v}' - \dot{u}'v' - u'\dot{v}' + \dot{\phi}w'$$

The gravity vector is represented in R_0 by:

$${}^{0}\overrightarrow{g} = [0, 0, -g]^T$$

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2. Dynamics

Kinetic Energy:

The kinetic energy T of the system is the sum of kinetic energies of the base, the link and the payload linearized to the second order.

$$T = T_B + T_l + T_p$$

Where

$$T_B = \frac{1}{2} I_B \dot{\theta}^2$$

I_B is the base inertia about the Z_0 axis.

$$T_l = \frac{1}{2} \iiint_V v(P/0)^2 dm$$

$$T_l = \frac{\rho}{2}\Bigg\{\pi R^2 \int_0^L (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)dx + \frac{1}{4}\pi R^4 \int_0^L (\dot{v}'^2 + \dot{w}'^2 + 2\dot{\phi}^2)dx + \dot{\theta}^2 \Bigg[\frac{1}{3}\pi R^2 L^3 + \frac{1}{4}\pi R^4 L$$

$$+ \pi R^2 \int_0^L (u^2 + v^2)dx + \frac{1}{4}\pi R^4 \int_0^L w'^2 dx + 2\pi R^2 \int_0^L xu\,dx - \frac{1}{2}\pi R^2 \int_0^L (L^2 - x^2)(v'^2 + w'^2)dx\Bigg] + 2\dot{\theta}\Bigg[\pi R^2 \int_0^L x\dot{v}\,dx$$

$$- \frac{1}{4}\pi R^4 \int_0^L (-\dot{v}' + \dot{u}'v' + u'\dot{v}' - 2w'\dot{\phi})dx + \pi R^2 \int_0^L (u\dot{v} - \dot{u}v)dx\Bigg]\Bigg\}$$

ρ is the mass density of the beam that is considered homogeneous

$$T_p = \frac{1}{2}\overrightarrow{\Omega_{p/0}}.I_p\overrightarrow{\Omega_{p/0}} + \frac{1}{2}m_p v(C/0)^2$$

Where

$$I_p = \begin{bmatrix} I_1 & I_4 & I_5 \\ I_4 & I_2 & I_6 \\ I_5 & I_6 & I_3 \end{bmatrix}$$

Hence

$$T_p = \frac{1}{2}\Bigg[I_1\left(\phi_L^2 cos(\theta)^2 + w_L'^2 sin(\theta)^2 + 2\phi_L w_L' cos(\theta)sin(\theta)\right) + I_2\left(\phi_L^2 sin(\theta)^2 + w_L'^2 cos(\theta)^2 - 2\phi_L w_L' cos(\theta)sin(\theta)\right) + I_3\left(\theta^2 + \dot{v}_L'^2 + 2\theta(\dot{v}_L' - \dot{u}_L' v_L' - u_L' \dot{v}_L' + \phi_L w_L')\right)$$

$$+ 2I_4\left((\dot{\phi}_L^2 - w_L'^2)cos(\theta)sin(\theta) - \dot{\phi}_L w_L'(2cos(\theta)^2 - 1)\right) + 2I_5\left(\theta\left((\dot{\phi}_L + \frac{1}{2}(v_L' w_L' - \dot{v}_L' w_L'))cos(\theta) - (-w_L' + \dot{u}_L' w_L' + u_L' w_L' + \dot{v}_L' \dot{\phi}_L)\sin(\theta)\right) + \dot{v}_L' \dot{\phi}_L cos(\theta) + \dot{v}_L' w_L' sin(\theta)\right)$$

$$+ 2I_6\left(\dot{\theta}\left((\dot{\phi}_L + \frac{1}{2}(v_L' w_L' - \dot{v}_L' w_L'))sin(\theta) + (-w_L' + \dot{u}_L' w_L' + u_L' w_L' + \dot{v}_L' \dot{\phi}_L)\cos(\theta)\right) + \dot{v}_L' \dot{\phi}_L sin(\theta) - \dot{v}_L' w_L' cos(\theta)\right)\Bigg] + \frac{1}{2}mp\Bigg[\dot{u}_L^2 + \dot{v}_L^2 + w_L^2 + c^2(\dot{v}_L'^2 + w_L'^2)$$

$$+ 2c(\dot{v}_L \dot{v}_L' + w_L w_L') + \theta^2\left(L^2 + u_L^2 + v_L^2 + c^2(1 - w_L'^2) + 2L[u_L - \frac{1}{2}\int_0^L (v'^2 + w'^2)ds + c(1 - \frac{1}{2}(v_L'^2 + w_L'^2))] + 2c(u_L + v_L v_L') - c\int_0^L (v'^2 + w'^2)ds\right)$$

$$+ 2\theta\left((L + c)(\dot{v}_L + c(\dot{v}_L' - \dot{u}_L' v_L' - u_L' \dot{v}_L')) + u_L(\dot{v}_L + c\dot{v}_L') - \dot{u}_L(v_L + cv_L')\right)\Bigg]$$

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Potential Energy:

The potential energy V of the system is the sum of potential energies of the base, the link and the payload linearized to the second order.

The potential energy V_B of the base which is its gravitational potential energy equals a constant C_B

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The potential energy of the link is the sum of its gravitational potential energy and its strain energy:

$$V_l = V_{gravit} + V_{str}$$

V_{gravit} is the gravitational potential energy of the link that equals:

$$V_{gravit} = -\int_{r=0}^{R}\int_{\gamma=0}^{2\pi}\int_{x=0}^{L} \vec{g}\,\overrightarrow{O_0P}\rho r dr d\gamma dx = \rho g \pi R^2 \int_{x=0}^{L} w dx$$

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V_{str} is the strain energy of the link and it is the sum of strain energies due to different strains:

$$V_{str} = V_u + V_v + V_w + V_\phi$$

Where

$$V_u = \frac{1}{2}\iiint_V E u'^2 dV = \frac{1}{2}\pi R^2 E \int_0^L u'^2 dx \qquad V_v = \frac{1}{2}\iiint_V E v''^2 y^2 dV = \frac{1}{8}\pi R^4 E \int_0^L v''^2 dx \qquad V_w = \frac{1}{2}\iiint_V E w''^2 z^2 dV = \frac{1}{8}\pi R^4 E \int_0^L w''^2 dx$$

$$V_\phi = \frac{1}{2}\iiint_V G r^2 \phi'^2 dV = \frac{1}{4}\pi R^4 G \int_0^L \phi'^2 dx$$

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E and G are the young modulus and the shear modulus of the beam material respectively.

The potential energy of the payload is its gravitational potential energy that equals:

$$V_p = -m_p\,\vec{g}\,\overrightarrow{O_0C} = m_p g\left(w_L + c(w'_L - u'_L w'_L)\right)$$

3.Rayleigh Dissipation Function

Rayleigh dissipation function R is the expression of the energy dissipated due to motor friction and internal damping effect of each deformation (u, v, w, ϕ), the dissipation is based on the Kelvin-Voigt model whose expression is given by:

$$\mathcal{R} = \mathcal{R}_{mot} + \mathcal{R}u + \mathcal{R}v + \mathcal{R}w + \mathcal{R}_{\phi}$$

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Where

$$\mathscr{R}_{mot} = \frac{1}{2} b_m \dot{\theta}^2$$

$$\mathscr{R}_u = \frac{1}{2} \iiint_V \sigma_u^d \epsilon_u dV = \frac{1}{2} \pi R^2 C_X \int_{x=0}^{L} u'^2 dx \quad \mathscr{R}_v = \frac{1}{2} \iiint_V \sigma_v^d \epsilon_v dV = \frac{1}{8} \pi R^4 C_Y \int_{x=0}^{L} v''^2 dx \quad \mathscr{R}_w = \frac{1}{2} \iiint_V \sigma_w^d \epsilon_w dV = \frac{1}{8} \pi R^4 C_Z \int_{x=0}^{L} w''^2 dx$$

$$\mathscr{R}_\phi = \frac{1}{2} \iiint_V \tau_\phi^d \gamma_\phi dV = \frac{1}{4} \pi R^4 C_\Phi \int_{x=0}^{L} \phi'^2 dx$$

The positive constant b_m is the motor viscous friction coefficient.

C_x, C_y, C_z are the internal damping coefficients along the x axis, the y axis, the z axis respectively, and C_Φ is the torsional internal damping coefficient.

4.Motion equations

The utilization of extended Hamilton principle yields the motion equation and boundary conditions.

$$0 = \int_{t_1}^{t_2} (\delta T - \delta V + T_{mot}\, \delta\theta + \delta\zeta)dt$$

where δζ is the variation of work done by the dissipative forces whose expression is derived from Rayleigh dissipation function as follows

$$\mathscr{R} = \frac{1}{2} \iiint_V \sigma^d \, \dot{\epsilon} \, dV \quad \Rightarrow \quad \delta\zeta = - \iiint_V \sigma^d \, \delta\epsilon \, dV$$

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The beam is clamped at the joint:

$$u(0,t) = v(0,t) = w(0,t) = \phi(0,t) = 0 \quad v'(0,t) = w'(0,t) = 0$$

The dynamic equation associated with θ is :

$$\begin{aligned}
T_{mot} = {} & b_m \dot{\theta} + \frac{1}{2} I_B \ddot{\theta} - \Big[I_1 \Big(cos(\theta) sin(\theta) (w_L'^2 - \dot{\phi}_L^2) + \dot{\phi}_L w_L' (2cos(\theta)^2 - 1) \Big) + I_2 \Big(cos(\theta) sin(\theta) (\dot{\phi}_L^2 - w_L'^2) - \dot{\phi}_L w_L' (2cos(\theta)^2 - 1) \Big) + \\
& I_4 \Big((\dot{\phi}_L^2 - \dot{w}_L'^2)(2cos(\theta)^2 - 1) + 4 \dot{\phi}_L \dot{w}_L' cos(\theta) sin(\theta) \Big) + \dot{\theta} [(\dot{\phi}_L + \frac{1}{2} (v_L' \dot{w}_L' - \dot{v}_L' w_L'))(-I_5 sin(\theta) + I_6 cos(\theta)) + (-\dot{w}_L' + \dot{u}_L' w_L' + u_L' \dot{w}_L' + v_L' \dot{\phi}_L)(-I_5 cos(\theta) - I_6 sin(\theta))] \\
& + \dot{v}_L' [\dot{\phi}_L (- I_5 sin(\theta) + I_6 cos(\theta)) + \dot{w}_L' (I_5 cos(\theta) + I_6 sin(\theta))] - \frac{\partial}{\partial t} \Big(I_3 (\dot{\theta} + \dot{v}_L' - \dot{u}_L' v_L' - u_L' \dot{v}_L' + \dot{\phi}_L w_L') + (\dot{\phi}_L + \frac{1}{2} (v_L' \dot{w}_L' - \dot{v}_L' w_L'))(I_5 cos(\theta) + I_6 sin(\theta)) \\
& + (-\dot{w}_L' + \dot{u}_L' w_L' + u_L' \dot{w}_L' + v_L' \dot{\phi}_L)(-I_5 sin(\theta) + I_6 cos(\theta)) + m_p \Big\{ \dot{\theta} \Big(L^2 + u_L^2 + v_L^2 + c^2 (1 - w_L'^2) + 2L [u_L - \frac{1}{2} \int_0^L (v'^2 + w'^2) ds + c (1 - \frac{1}{2} (v_L'^2 + w_L'^2))] + 2c (u_L + v_L v_L') \\
& - c \int_0^L (v'^2 + w'^2) ds \Big) + (L + c)(\dot{v}_L + c(\dot{v}_L' - \dot{u}_L' v_L' - u_L' \dot{v}_L')) + u_L (\dot{v}_L + c \dot{v}_L') - \dot{u}_L (v_L + c v_L') \Big\} \Big) \Big] + \frac{\rho}{2} \Big\{ \frac{\partial}{\partial t} \Big(2 \dot{\theta} \Big[\frac{1}{3} \pi R^2 L^3 + \frac{1}{4} \pi R^4 L + \pi R^2 \int_0^L (u^2 + v^2) dx \\
& + \frac{1}{4} \pi R^4 \int_0^L w'^2 dx + 2 \pi R^2 \int_0^L x u dx - \frac{1}{2} \pi R^2 \int_0^L (L^2 - x^2)(v'^2 + w'^2) dx \Big] \Big) + 2 \frac{\partial}{\partial t} \Big(\pi R^2 \int_0^L x \dot{v} dx - \frac{1}{4} \pi R^4 \int_0^L (-\dot{v}' + \dot{u}' v' + u' \dot{v}' - 2 w' \dot{\phi}) dx + \pi R^2 \int_0^L (u \dot{v} - \dot{u} v) dx \Big) \Big\}
\end{aligned}$$

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The equation satisfied by u:

$$0 = \frac{\rho}{2}\left(-2\pi R^2 \ddot{u} + 2\pi R^2 \theta^2 u + 2\pi R^2 \theta^2 x + \pi R^2 (4\dot{\theta}\dot{v} + 2\ddot{\theta} v) - \frac{1}{2}\pi R^4 \ddot{\theta} v'' \right) + \pi R^2 C_X \dot{u}'' + \pi R^2 E u''$$

The equation satisfied by v:

$$0 = \frac{\rho}{2}\left(-2\pi R^2 \ddot{v} + \frac{1}{2}\pi R^4 \ddot{v}'' + 2\pi R^2 \theta^2 v - \pi R^2 \theta^2 (2xv' + (x^2 - L^2)v'') - \pi R^2 (4\dot{\theta}\dot{u} + 2\ddot{\theta} u + 2x\ddot{\theta}) - \frac{1}{2}\pi R^4 \ddot{\theta} u'' \right)$$
$$- \frac{1}{4}\pi R^4 C_Y \dot{v}'''' - \frac{1}{4}\pi R^4 E v'''' + m_p \theta^2 (L + c) v''$$

The equation satisfied by w:

$$0 = \frac{\rho}{2}\left(-2\pi R^2 \ddot{w} + \frac{1}{2}\pi R^4 \ddot{w}'' - \frac{1}{2}\pi R^4 \dot{\theta}^2 w'' - \pi R^2 \dot{\theta}^2 (2xw' + (x^2 - L^2)w'') - \pi R^4 \dot{\theta}\dot{\phi}' \right) - \frac{1}{4}\pi R^4 C_Z \dot{w}'''' + m_p \dot{\theta}^2 (L + c) w''$$
$$- \frac{1}{4}\pi R^4 E w'''' - \rho g \pi R^2$$

The equation satisfied by Φ:

$$0 = \frac{\rho}{2}\left(-\pi R^4 \ddot{\phi} - \pi R^4 (\ddot{\theta} w' + \dot{\theta}\dot{w}') \right) + \frac{1}{2}\pi R^4 C_\Phi \dot{\phi}'' + \frac{1}{2}\pi R^4 G \phi''$$

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Since the beam at x = L, has a free end the boundary conditions are:

$$0 = -\pi R^2 C_X \dot{u}'_L + \frac{\rho}{4}\pi R \ddot{\theta} v'_L - \pi R^2 E u'_L - \frac{\partial}{\partial t}\left[mp\left(\dot{u}_L - \dot{\theta}(v_L + c v'_L)\right)\right] + mp\left[\dot{\theta}^2(u_L + L + c) + \dot{\theta}(\dot{v}_L + c\dot{v}'_L)\right]$$

$$0 = -\frac{\partial}{\partial t}\left[-I_3\dot{\theta}\dot{v}'_L + w'_L(-I_5\dot{\theta}sin(\theta) + I_6\dot{\theta}cos(\theta)) - m_p\dot{\theta}c(L+c)\dot{v}'_L\right] - I_3\dot{\theta}\dot{v}'_L - I_5\dot{\theta}\dot{w}'_L sin(\theta) + I_6\dot{\theta}cos(\theta)\dot{w}'_L - m_p\dot{\theta}c(L+c)\dot{v}'_L + m_p g c w'_L$$

$$0 = \frac{1}{4}\pi R^4 C_Y \dot{v}'''_L - \frac{\rho}{4}\pi R^4 \ddot{v}'_L + \frac{\rho}{4}\pi R^4 \ddot{\theta}(u'_L - 1) + \frac{1}{4}\pi R^4 E v'''_L - \frac{\partial}{\partial t}\left[mp\left(\dot{v}_L + c\dot{v}'_L + \dot{\theta}(L + c + u_L)\right)\right] + mp\left(\dot{\theta}^2(v_L + c v'_L) - \dot{\theta}\dot{u}_L - \dot{\theta}^2(L+c)v'_L\right)$$

$$0 = -\frac{1}{4}\pi R^4 C_Y \dot{v}''_L - \frac{1}{4}\pi R^4 E v''_L - \frac{\partial}{\partial t}\left[I_3\left(\dot{v}'_L + \dot{\theta}(1 - u'_L)\right) + I_5\left(-\frac{1}{2}\dot{\theta}w'_L cos(\theta) + \dot{\phi}_L cos(\theta) + \dot{w}'_L sin(\theta)\right) + I_6\left(-\frac{1}{2}\dot{\theta}w'_L sin(\theta) + \dot{\phi}_L sin(\theta) - \dot{w}'_L cos(\theta)\right)\right.$$
$$\left. + m_p\left(c^2\dot{v}'_L + c\dot{v}_L + \dot{\theta}c(L+c)(1 - u'_L) + \dot{\theta}c u_L\right)\right] - I_3\dot{\theta}\dot{u}'_L + I_5\left(\frac{1}{2}\dot{\theta}\dot{w}'_L cos(\theta) - \dot{\theta}\dot{\phi}_L sin(\theta)\right) + I_6\left(\frac{1}{2}\dot{\theta}\dot{w}'_L sin(\theta) + \dot{\theta}\dot{\phi}_L cos(\theta)\right) + m_p\left(\dot{\theta}^2(-Lcv'_L + cv_L) - \dot{\theta}c(L+c)\dot{u}'_L - \dot{\theta}c\dot{u}_L\right)$$

$$0 = \frac{1}{4}\pi R^4 C_Z \dot{w}'''_L + \frac{1}{4}\pi R^4 E w'''_L - m_p g - \frac{\rho}{4}\pi R^4 \ddot{w}'_L + \frac{\rho}{4}\pi R^4 \dot{\theta}^2 w'_L + \frac{\rho}{2}\pi R^4 \dot{\theta}\dot{\phi}_L - m_p(\ddot{w}_L + c\ddot{w}'_L) - m_p\dot{\theta}^2(L+c)w'_L$$

$$0 = -\frac{1}{4}\pi R^4 C_Z \dot{w}_L'' - \frac{1}{4}\pi R^4 E w_L'' - m_p g c(1 - u_L') - \frac{\partial}{\partial t}\Big[I_1\Big(sin(\theta)^2 \dot{w}_L' + cos(\theta)sin(\theta)\dot{\phi}_L\Big) + I_2\Big(cos(\theta)^2 \dot{w}_L' - cos(\theta)sin(\theta)\dot{\phi}_L\Big) + I_4\Big(-2\dot{w}_L' cos(\theta)sin(\theta) - \dot{\phi}_L(2cos(\theta)^2 - 1)\Big)$$
$$+ I_5\Big(\frac{1}{2}\theta \dot{v}_L' cos(\theta) + sin(\theta)(\dot{v}_L' - \dot{\theta}(u_L' - 1))\Big) + I_6\Big(\frac{1}{2}\theta \dot{v}_L' sin(\theta) + cos(\theta)(-\dot{v}_L' + \dot{\theta}(u_L' - 1))\Big) + m_p\Big(c^2 w_L' + c w_L\Big)\Big] + I_3\theta\dot{\phi}_L + I_5\Big(-\frac{1}{2}\theta\dot{v}_L' cos(\theta) - \theta\dot{u}_L' sin(\theta)\Big)$$
$$+ I_6\Big(-\frac{1}{2}\theta\dot{v}_L' sin(\theta) + \theta\dot{u}_L' cos(\theta)\Big) - m_p\theta^2 c(L + c)w_L'$$

$$0 = -\frac{1}{2}\pi R^4 C_\Phi \dot{\phi}_L' - \frac{1}{2}\pi R^4 G\phi_L' - \frac{\partial}{\partial t}\Big[I_1\Big(cos(\theta)^2 \dot{\phi}_L + cos(\theta)sin(\theta)\dot{w}_L'\Big) + I_2\Big(sin(\theta)^2 \dot{\phi}_L - cos(\theta)sin(\theta)\dot{w}_L'\Big) + I_4\Big(2cos(\theta)sin(\theta)\dot{\phi}_L - \dot{w}_L'(2cos(\theta)^2 - 1)\Big)$$
$$+ I_5\Big((\theta + \dot{v}_L')cos(\theta) - \theta\dot{v}_L' sin(\theta)\Big) + I_6\Big((\theta + \dot{v}_L')sin(\theta) + \theta\dot{v}_L' cos(\theta)\Big)\Big]$$

u, v, ϕ must also satisfy these conditions:

$$u(x,0) = \lim_{t\to\infty} u(x,t) = 0$$

$$v(x,0) = \lim_{t\to\infty} v(x,t) = 0, \phi(x,0) = \lim_{t\to\infty} \phi(x,t) = 0$$

and w must satisfy

$$w(x,0) = \lim_{t\to\infty} w(x,t) = \tilde{w}(x)$$

Where

$$\tilde{w}(x) = \int_0^x tan\left(\frac{l(2a-l)}{2b}\right) dl$$

With

$$a = L - \delta$$

$$b = \frac{EI}{F}$$

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F is the weight of the payload and equals $m_p g$.

I is the second moment of area of the beam that have a circular cross-section and equals:

$$I = \iint y^2 dydz = \int_{r=0}^{R} \int_{\gamma=0}^{2\pi} r^2 cos(\gamma)^2 r dr d\gamma = \frac{\pi R^4}{4}$$

δ is the foreshortening term due to the bending of the beam, expressed by:

$$\delta = -\frac{1}{2} \int_0^L \tilde{w}'^2(x)dx$$

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- Discussion

The motion equations are decoupled when the motor rotate with constant angular velocity

$$\dot{\theta} = \Omega$$

The equation satisfied by v yields $\dot{u} = L_1\,(v)$

Taking the time derivative of the equation satisfied by u and using the last equation yields the PDE verified by v That can be expressed by :

$$L_2\,(v) = 0$$

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The equation satisfied by w yields

$$\dot{\phi}' = L_3(w) + c$$

Taking both time and spatial derivatives of the equation satisfied by ϕ and using the last equation yields the PDE verified by w that can be expressed by:

$$L_4\,(w) = 0$$

Where L_1, L_2, L_3, L_4 are linear operators.

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The mathematical problem:

Solving the previous partial differential equations with coupled boundary condition.

The quest is to find a numerical method that yields stable solutions

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- Conclusion

The single-link flexible manipulator's mechanical modeling relies on solving the coupled PDE of motion. Once the solutions are established, the mechanical modeling could be extended to flexible serial link manipulators, which will allow the development of new techniques for robust control of their movements.

Thank you for your attention