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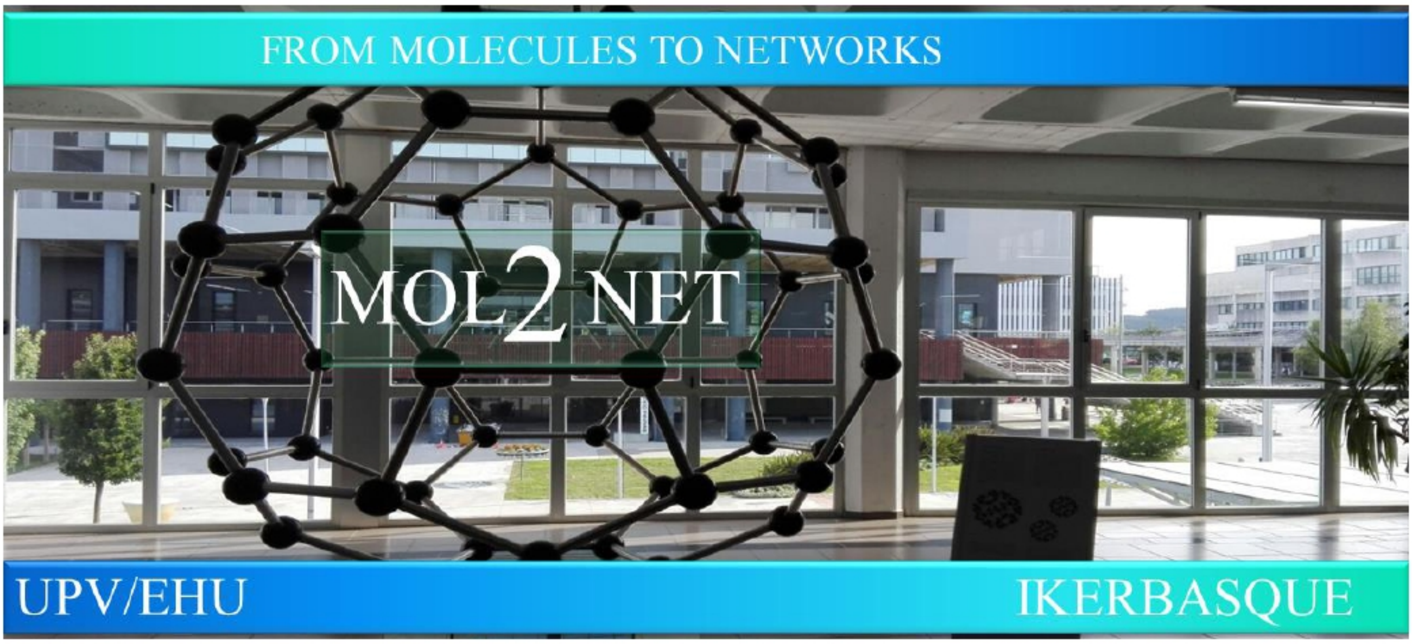

Optimal packing of convex polygons defined by their vertices

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Abstract

In optimal packing problems, there is a set of small elements (load) to be arranged in one or more large objects (containers), fulfilling the non-overlapping conditions between the small components and the containment conditions (the load does not exceed the dimensions of the container), in addition, there is an objective to optimize.

The main objective of this investigation is to find acceptable solutions in a reasonable time to the instances of the problem. of packing convex polygons in convex containers (circular and circular sections) of variable dimensions, using an exact mathematical nonlinear programming model, defining polygons or items/loads by their vertices and using the Lagrangian approach and convexity conditions. In addition to determining the effect on the packaging, having as control parameters the number of elements to be packaged, the type of element, and the type of container.

Keywords: Optimal packing, convex polygons.

1. Introduction

Optimal packing problems (OPP) have been of great value in recent decades as a research topic. Its wide range of applications in practical and theoretical areas, make it a subject with great presence in the literature on topics ranging from container loading in logistics(Truong et al., 2020), in the paper, glass, steel, and agriculture industries (Zhou et al., 2019), in areas vital such as medicine (radio-surgical treatments) and physics (approximation of granular materials and particle packaging (Gan and Yu, 2020)), to current issues and part of the industry 4.0 project such as additive manufacturing known as 3D printing (Romanova et al., 2019; Zhao et al., 2020; Oh et al., 2018).

In OPP, there is a set of small elements called cargo, which must be placed or assigned to one or more large objects called containers. In addition, it must comply with the non-overlap between small elements, and that said set of elements does not exceed the dimensions of the container to which they were assigned. The classification of these problems is NP-hard Fowler et al. (1981) due to their computational complexity.

This research will focus on solving the problem of packing an arbitrary assortment of small convex elements in a single convex container of variable dimensions that can be treated as the open(s) dimension(s) problem (ODP).

The ODP is a problem where the set of elements must be fully accommodated in a large object or container. The container, having at least one dimension, can be considered a variable. In other words, this problem involves a decision about setting the extent to the variable dimensions of the container, as well as the input value or other measure such as length, size or volume should be minimized (Wascher et al., 2007).

This work presents an experimental design for a non-linear model with a Lagrangian approach for non-intersection conditions that describes the figures by their vertices presented in

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Martínez-Noa (2020). In addition, he analyzes how the type of figure influences the packing density as a factor in the quality of the result.

Its structure is as follows. In section 2 the mathematical model for two types of containers is presented. Next, section 3 shows a description of the instances. In section 4 the design of the experiment is presented. In section refSection5 some of the computational results are shown. In section ref Section6 a statistical analysis of the results is performed for the container type circular and type circular sections. Finally, section ref Section7 shows the conclusions reached.

2. Mathematical model

This section describes the mathematical model presented in Martínez-Noa (2020) where the type of container is varied (circular and circular- section). This model uses the no interception and containment conditions presented in Litvinchev et al. (2020).

2.1. Circular section container

Declaration of variables:

- **i** Set of polygons.
- **j** Set of vertices.
- **k** Dimensions.
- **R** Container radius.
- $\alpha_{i,h}$ Lagrange multiplier.
- $\beta_{i,h}$ Lagrange multiplier.
- $v_{i,h,k}$ Lagrange multiplier.
- $\lambda_{i,h,j}$ Polytope convex combination factor i.
- $\mu_{i,h,j}$ Polytope convex combination factor h
- $\mathbf{x}_{i,j,k}$ Decision variable.
- $D(i, j, j+1)$ Distance from point j to point $j+1$ in polygon i
- σ_j Angle between adjacent sides.

equation 1 represents the objective function of the packing model in the container with the shape of a circular section of minimum radius.

$$Z = Min \quad R \tag{1}$$

S.t

Equations (2) and (3) are the geometric conditions that preserve the shape of the polygons to be packed. In the equation (2) $D(i, j, j+1)^2$ is the distance in the figure i between the points j and $j+1$.

$$\sum_k \left(x_{ijk} - x_{ij+1k}\right)^2 = D(i, j, j+1)^2 \tag{2}$$

$$\sum_k \left(x_{ij-1k} - x_{ijk}\right)\left(x_{ij+1k} - x_{ijk}\right) = D(i, j-1, j) \cdot D(i, j, j+1) \cdot \cos\sigma_{j-1} \quad \forall i, 1 < j < |J| \tag{3}$$

equations(4),(5),(6),(7),(8),(9)y(10) guarantee no intersection between polygons.

$$\alpha_{ih} + \beta_{ih} < 2, \quad i, h \in I, \quad h > i \tag{4}$$

$$\sum_j \left(\lambda_{ihj} + \mu_{ihj}\right) = \alpha_{ih} + \beta_{ih}, \quad \forall i, h, \quad h > i \tag{5}$$

$$\sum_j \sum_k \lambda_{ihj} \cdot x_{ijk} = \sum_j \sum_k \mu_{ihj} \cdot x_{ijk}, \quad \forall i, h, \quad h > i \tag{6}$$

$$1 + \sum_k v_{i,h,k} \cdot x_{i,j,k} - \alpha_{i,h} \leq 0, \quad \forall i, h, \quad h > i, \quad \forall j \tag{7}$$

$$1 - \sum_k v_{ihk} \cdot x_{hjk} + \beta_{i,h} \leq 0, \quad \forall i, h, \quad h > i, \quad \forall j \tag{8}$$

$$\sum_j \lambda_{ihj} \leq 1 \quad \forall ih, \quad h > i \tag{9}$$

$$\sum_j \mu_{ihj} \leq 1 \quad \forall i, h, \quad h > i \tag{10}$$

constraint (11) guarantees that the polygons are contained in the circular section that the container represents.

$$x_{ij1}^2 + x_{ij2}^2 \leq R^2 \tag{11}$$

constraint (12) the non-negative nature of the variables.

$$x_{ijk}, v \in \mathbb{R}s, \alpha_{ih}, \beta_{ih}, \lambda_{ihj}, \mu_{ihj}, R \geq 0 \quad \forall i, j, k \tag{12}$$

2.2. *Circular container*

For the case of the circular container, only change the equation 11 by

$$(x_{ij1} - 30)^2 + (x_{ij2} - 30)^2 \leq R^2, \tag{13}$$

is due to the non-negativity constraint of the variables, so the center of the circumference is shifted from the origin of coordinates $(0; 0)$ to the point $(30; 30)$.

3. Description of the instances

The instances analyzed in this research were created with an instance generator programmed in Python language (Python Core Team , 2020), to validate the model proposed. The table 1 shows the number of instances for the two types of containers analyzed (section-circular, circular). The instances for each figure only vary in the number of items (5, 6, 7, 8, 9, 10) to pack. In the case of pentagons, there are ten instances for the regular ones (pentagons$_r$) and the same amount for the irregular ones (pentagons$_i$).

Table 1: Number of instances per figure

Figures	Number of instances
triangles	6
rectangles	6
squares	6
pentagons$_r$	6
pentagons$_i$	6
mixed quadrilaterals	6
hexagons	6

4. Design of experiment

Two types of containers and seven types of different figures to be packed with six different instance sizes are contemplated in this experimentation. It used a complete factorial design with three control factors per treatment, so there are 84 treatments. These control factors are:

- Two-level container type (section-circular, circular),
- Types of seven-level figures (triangles, squares, rectangles, pentagons (regular and irregular) and hexagons, mixed quadrilaterals),
- Number of items in the instance (5, 6, 7, 8, 9, 10).

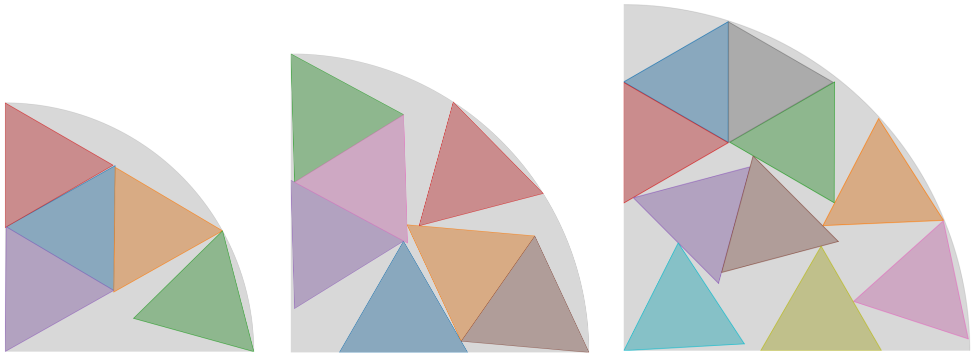
5. Computational results

5.1. Hardware and software used

All experiments were running on a Dell Latitude 5580, RAM 16Gb 256Gb Ssd Fhd 15.6 with Intel (R) Core (TM) i7–7600, CPU @ 3.4Ghz, 3201Mhz, 4 cores, 8 threads using Windows 10 professional and AMPL Modeling language. Were solving nonlinear optimization problems with the global solver BARON(Branch & Reduce Optimization Navigator) for AMPL version 19.12.7

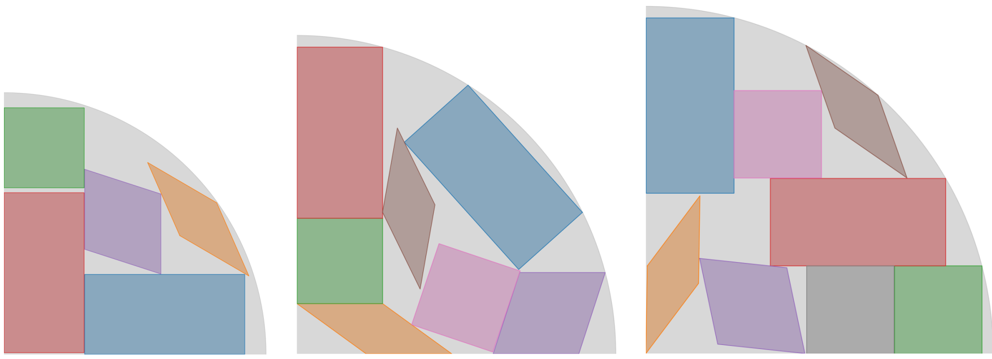
5.2. Results

Figures (1–6) show some of the computational results obtained in experimentation, the challenge of data and images are available in Martínez-Noa (2020).



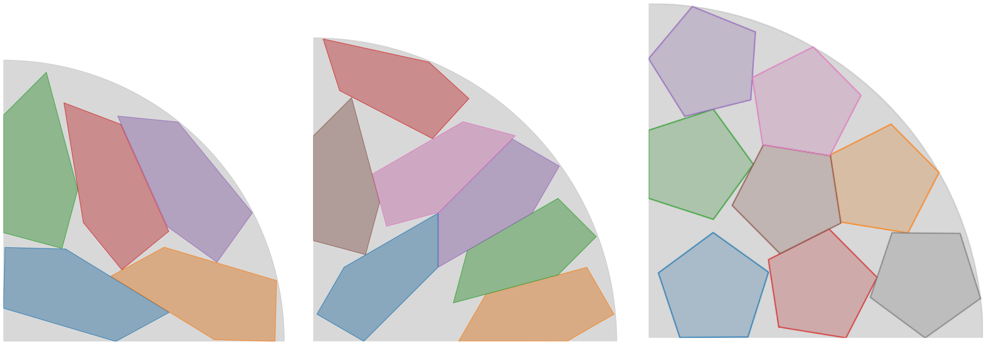
(b) 5, radio = 1.9937 (c) 7, radio = 2.2834 (d) 10, radio = 2.8688

Figure 1: Results of triangles in container circular section



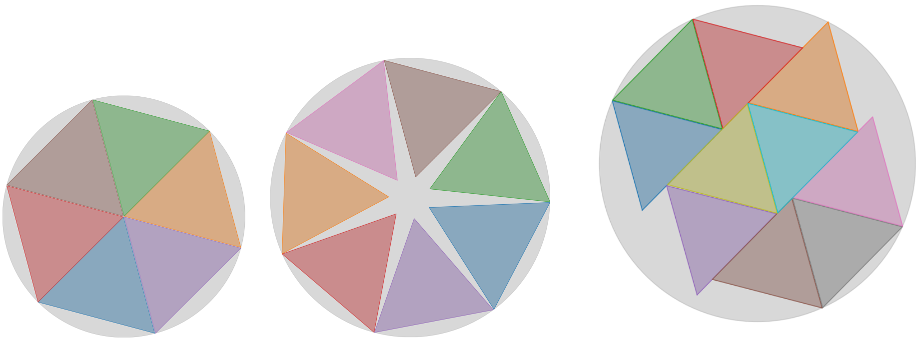
(a) 5, radio = 3.27443 (b) 7, radio = 3.72475 (c) 8, radio = 3.96078

Figure 2: Results of mixed quadrilaterals in container circular section



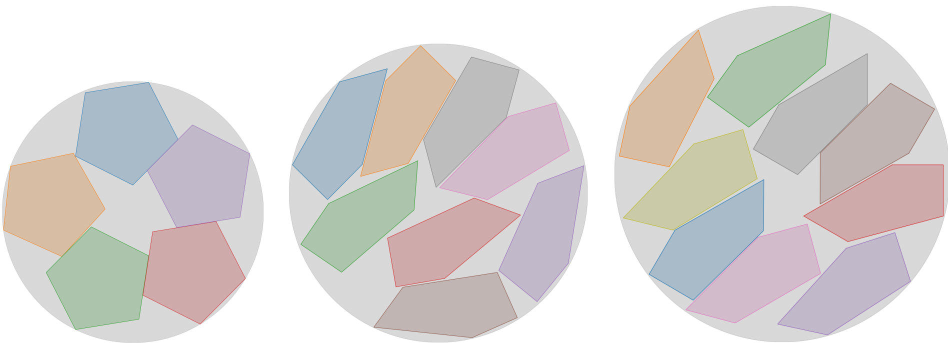
(a) 5, radio = 4.63061 (b) 7, radio = 5.6060 (c) 8, radio = 4.9109

Figure 3: Results of identical pentagons in circular section container



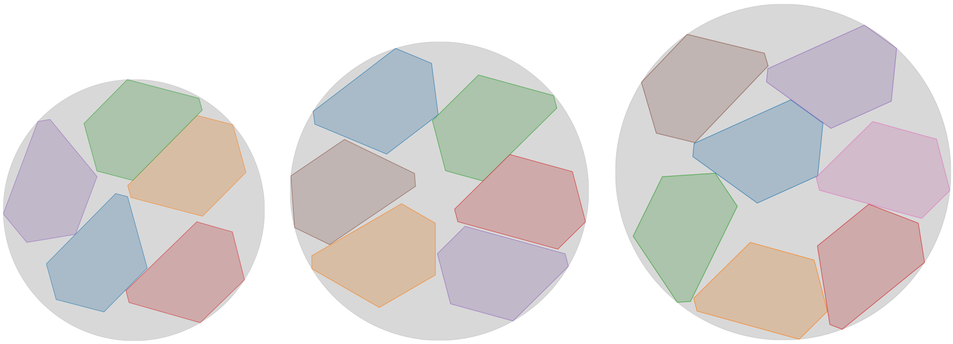
(a) 6,radio = 1.00424 (b) 7,radio =1.15669 (c) 10,radio = 1.43058

Figure 4: Results of identical triangles in circular container



(a) 5, radio = 2.0354 (b) 8, radio = 3.02036 (c) 9, radio = 3.28145

Figure 5: Identical pentagons results in circular container



(a) 5, radio = 2.6486 (b) 6, radio = 2.8790 (c) 7, radio = 3.1664

Figure 6: Identical hexagon results in a circular container

6. Statistical analysis of the results

Part of this investigation is to determine if shape type influences packing density. So a statistical analysis of the data is carried out, dividing them by container types.

6.1. Circular section container

We want to use an ANOVA (one-way analysis of variance) for analysis of the results, taking as a dependent variable the percentage of occupancy and as a factor the type of figure.

Three assumptions are verified before performing the ANOVA (one-way analysis of variance): the populations (probability distributions of the dependent variable corresponding to the factor) are normals, the K samples on which the treatments are applied are independent, and the populations have equal variance (homoscedasticity) (Humberto, 2008).

The Shapiro-Wilk test is performed to check the first assumption, which calculates a statistical W that tests whether a random sample $x_1, x_2, ..., x_n$ comes from a distribution normal. The result of this test with a $W = 0.954$ and $p - value = 0.058$ greater than $alpha = 0.050$ shows that there is not enough evidence to reject the null hypothesis, which states that the dependent variable (density) follows a normal distribution. This is stated with a 95 % confidence interval. In Figure 7 it is shown that all points fall approximately along the reference line we can assume normality.

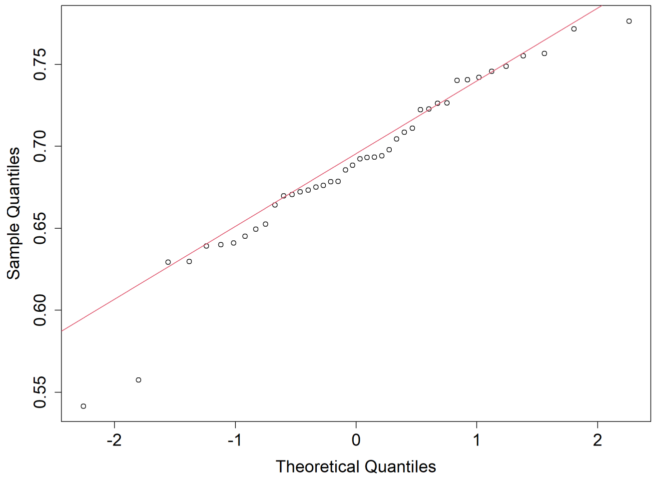


Figure 7: Normal Q-Q plot for the density variable

Given that the data are at the limit to accept that it is distributed normally, Fisher's and Bartlett's tests are not recommended because they are more sensitive to the lack of normality. Instead, it is better to use a test based on the median Levene test or the Fligner-Killeen test. In such a test with $\chi^2 = 8.994$ and a p-value $= 0.109$ greater than $\alpha = 0.050$ shows that there is not enough evidence to reject the null hypothesis, so the dependent variable (density) has homoscedasticity with a 95 % confidence interval.

Due to the results of previous tests, its used ANOVA in this analysis. For the type of figure factor, the following question is posed: Is there a difference between the percentage of occupancy of the different types of figures? The answer to this question is obtained by testing the following hypotheses:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 \tag{14}$$

$$H_1 : \mu_i \neq \mu_j \quad \text{for some} \quad i \neq j \tag{15}$$

With the values of the test statistic $\mathbf{F} = 1.030$ and p-value $= 0.415$ greater than $\alpha = 0.050$ there is not enough evidence to reject the hypothesis H_0 that indicates that the non-existence of statistically significant differences between the groups with a confidence interval of 95%. This can be seen in Figure ref fig: boxplotssc1.

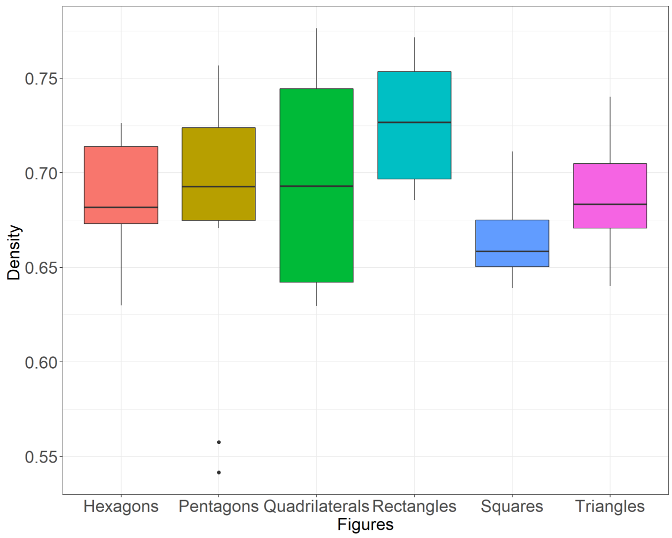


Figure 8: Boxplot that relates the types of figures and the % occupancy of the container

The results obtained for the case of the circular section container show that the type of figure does not significantly influence the density of the package.

6.2. Circular container

When performing the Shapiro-Wilk test, we obtain a **W** = 0.960 and a p-value = 0.150 greater than $\alpha = 0.050$ so there is enough evidence to reject the hypothesis that the dependent variable (density) follows a normal distribution with a 95% confidence interval. Figure 9 it is shown that the points are adjusted along the reference line, we can assume the normality of the variable.

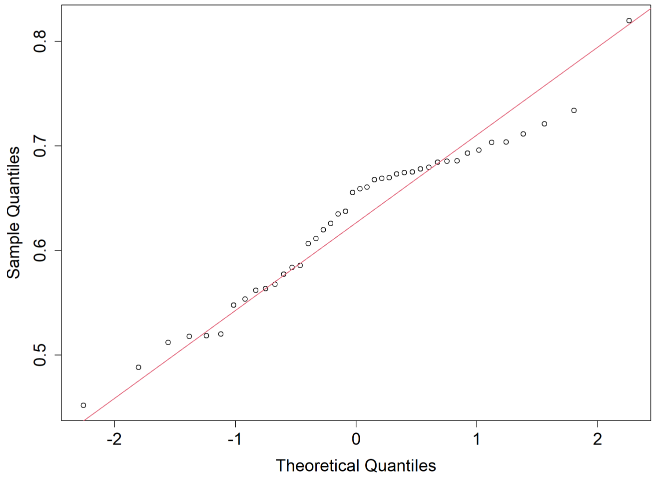


Figure 9: Normal Q-Q plot for the density variable

In the Fligner-Killeen test with $\chi^2 = 5.517$ and a p-value = 0.356 greater than $\alpha = 0.050$ shows that there is not enough evidence to reject the null hypothesis, for what the variable (density) has homoscedasticity with a confidence interval of 95%.

According to the data obtained in the ANOVA test, with a value of the test statistic **F** = 4.735 and p-value = 0.020 less than $\alpha = 0.050$, there is enough evidence to reject the null hypothesis, which indicates that there are statistically significant differences between the groups with a 95 % confidence interval. This can be seen in Figure 10.

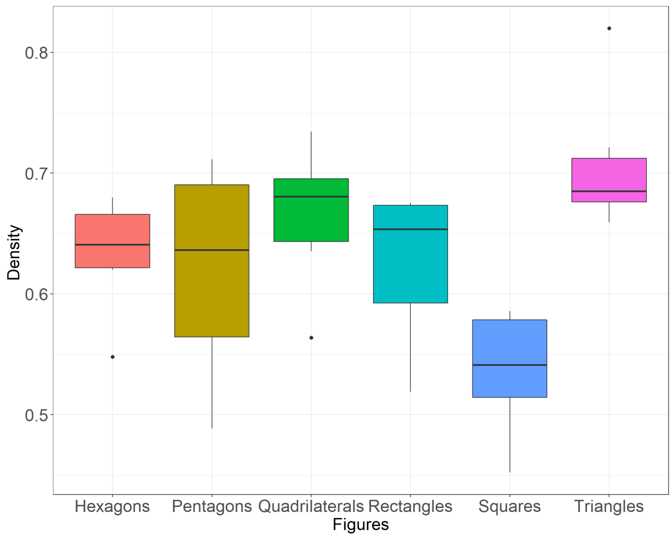


Figure 10: Boxplot that relates the types of figures and the % occupancy of the container

The ANOVA does not offer enough information to indicate between which groups the difference exists. Therefore, a Tukey HSD (Honestly Significant Difference) test is applied that shows the differences between the group means. The results of the test show that the hypothesis is rejected in the matching cases of the Squares-Quadrilaterals and Triangles-Squares instance, since the text p values is less than $alpha = 0.050$, as shown can be seen graphically in the figure 11.

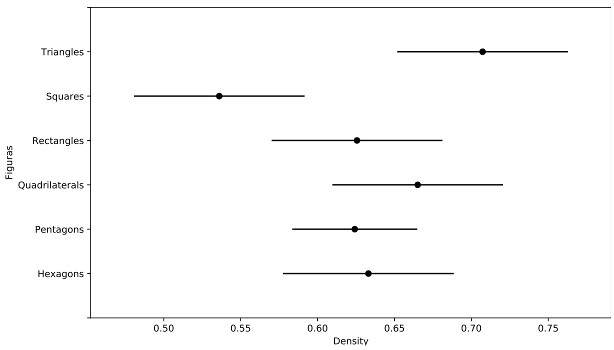


Figure 11: Simultaneous diagram that relates packing density, with the groups of the factor type of figures

7. Conclusions

The model used allows solving all the problems derived from the ODP class when you have convex polygons as elements to be packed in convex containers. It is easily scalable to **n** dimensions.

In the statistical analysis of the results of the computational experimentation, it was found that the type of figure does not influence the packing density for the circular section container with a 95 % confidence interval. On the other hand, for the circular container, the type of figure does influence the density with a 95 % confidence interval. There are differences between the square pairs - mixed quadrilaterals and squares - triangles.

Although the proposed model does not yield good solutions in reasonable times for medium and large instances, excellent results were obtained in the analyzed instances.

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