Entanglement – a higher order symmetry

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CONTENTS

Correlation Principle: In the case of two correlated events one measurement (observation) simultaneously yields at least two pieces of information.

- The state of a coin flip is is either (u, d) or (d, u) after an observation (post-observation):

$$|\psi\rangle = |u\rangle\,|d\rangle \qquad \text{or} \qquad |\psi\rangle = |d\rangle\,|u\rangle$$

- The state of a coin flip is a superposition of (u, d) and (d, u) prior to an observation (pre-observation):

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|u\rangle\,|d\rangle - |d\rangle\,|u\rangle)$$

  $-$ sign precludes both being in same state which is impossible by nature of coin.

- The pre-observation is a cloak for ignorance. Post-observation defines reality (classically).

ENTANGLED PAIRS

Analogous to the coin observations above (replacing "u" with "+" and "d" with "-" to indicate 1 and -1), consider the following two entangled (correlated) states defined over a Hilbert space, $L^2 \times H$:

1.
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|+\rangle + |-\rangle\,|-\rangle)$$

   One observation reduces this to either $|+\rangle\,|+\rangle$ or $|-\rangle\,|-\rangle$.

2.
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle)$$

   One observation reduces this to either $|+\rangle\,|-\rangle$ or $|-\rangle\,|+\rangle$.

ENTANGLED PAIRS

Main Result:

The two states above are also rotationally invariant:

- Given $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$ note that for $R \in SO(2)$

$$(R(\theta), R(\theta))|\psi\rangle = \frac{1}{\sqrt{2}}((R|+\rangle)(R|+\rangle) + (R|-\rangle)(R|-\rangle))$$
$$= \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle) = |\psi\rangle$$

- Similarly for the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle) \qquad (R(\theta), R(\theta))(|\psi\rangle) = |\psi\rangle$$

- We call these isotropically spin correlated (ISC) states.
- In addition, for singlet state $(R(\theta), R(\theta))(|\psi\rangle) = |\psi\rangle$ and $R \in SL(2, C)$.

A questions arises: are there ISC quantum states similar to the two above for $n \geq 3$? In other words, if such ISC states exist then they should have the following two properties (by definition):

1. For $n \geq 3$, in accorance with Einsteinian realism based on hidden parameters, prior to an observation the state should be of form

$$|\psi\rangle = \frac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_2 \ldots \mathbf{e}_n \pm \mathbf{e}_1^- \otimes \mathbf{e}_2^- \ldots \mathbf{e}_n^-] \tag{1}$$

2. On making an observation on $\mathbf{e}_i$ (for some i) the projection axiom of QM will reduce $|\psi\rangle$ to the state $|\psi\rangle_r$ given by

$$|\psi\rangle_r = \mathbf{e}_1 \otimes \mathbf{e}_2 \ldots \mathbf{e}_n \qquad \text{or} \qquad |\psi\rangle_r = \mathbf{e}_1^- \otimes \mathbf{e}_2^- \ldots \mathbf{e}_n^- \tag{2}$$

3. $|\psi\rangle$ should be rotationally invariant in order to maintain isotropy.

Theorem.

The Coupling principle: *Isotropically spin-correlated particles must occur in PAIRS.*

• It follows from the above theorem that there are two types of correlations

1. probability correlations generated by independent variables associated with individual states (hidden variables) that obey the Chapman-Kolmogorov equation.
2. probability correlations constituted by a subsistent relationship between states that violate the Chapman-Kolmogorov equation.

• It follows from the coupling principle that multi-particle systems can be divided into three categories:

1. Those containing coupled particles
2. Those containing decoupled (statistically independent) particles.
3. A mixture of both (para-statistics)

• A statistical analysis applied to completely indistinguisable particles reduces to Fermi-Dirac and Bose-Einstein statistics respectively.

Generalized spin-statistics Theorem

Theorem.

Let $V = V_1 \otimes \cdots \otimes V_n$, where each V_i is an n-dimensional vector space, and $T = T_1 \otimes \cdots \otimes T_n$ where for each i, j, $T_i = T_j$ and T_i is a linear operator on V_i. Let

$$\begin{aligned} v &\equiv v_1 \wedge v_2 \wedge \cdots \wedge v_n \\ &= \begin{pmatrix} v_{11} \\ \vdots \\ v_{n1} \end{pmatrix} \wedge \begin{pmatrix} v_{12} \\ \vdots \\ v_{n2} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} v_{1n} \\ \vdots \\ v_{nn} \end{pmatrix} \end{aligned}$$

then for $v \neq 0$

$$Tv = v \iff T \in \bigotimes_{1}^{n} SL(n, \mathcal{C}).$$

Fermi-Dirac statistics is invariant under the action of $SL(n, \mathcal{C})$.

PAULI'S SPIN STATISTICS THEOREM

1. In Pauli's paper, he first distinguishes between vectors and spinor representations of angular momentum, which he associates with tensors of even and odd rank respectively. He then uses the second quantization procedure to obtain a set of solutions, D and D_1, of a second order wave equation.
2. These latter, D_1 solutions, which are related to the Neumann's function and "the first Hankel cyclindrical function" ([**?**],p.721), are then carefully elimated as unphysical by invoking the postulate (nowadays refered to as micro-causality) that *"all physical quantities at finite distances exterior to the light cone (for* $|x_0' - x_0''| < |x' - x''|$*) are commutable.'*
3. The principle of micro-causality as interpreted by Pauli does not apply to entangled particles. There is no mention of entanglement in his paper on spin-statistics. It now remains to discuss the above mathematical results from the perspective of Pauli's article and in the overall context of the experimental evidence

- Now, consider a two particle system $|\psi(\lambda_1)\rangle \otimes |\psi(\lambda_2)\rangle \in \mathcal{S}_1 \otimes \mathcal{S}_2$ where $\mathcal{S}_1 = \mathcal{L}^2(\mathcal{R}^3) \otimes H_1$ and $\mathcal{S}_2 = \mathcal{L}^2(\mathcal{R}^3) \otimes H_2$ respectively. Note that each ket $|\psi(\lambda)\rangle \in \mathcal{S}$ can be written as $|\psi(q_1)\rangle \otimes s$, where s is a spinor.
- Also, let $\vec{S}_1 = (S_i(q_1), S_j(q_1), S_k(q_1))$ and $\vec{S}_2 = (S_i(q_2), S_j(q_2), S_k(q_2))$ be spin operators defined on the Hilbert spaces H_1 and H_2 respectively. We have already seen (Cor. 2) that Bose-Einstein statistics follow when *no* two states are ISC.
- It follows, trivially, that $[S_i(q_1) \otimes I_2, I_1 \otimes S_j(q_2)] = 0$, which means that in the case of Bose-Einstein statistics, spin operators must commute.

- On the other hand, in contrast to the triplet state, the singlet state defines a rotationally invariant state and obeys a Fermi-Dirac statistic. Once again, let $|\psi(\lambda_1, \lambda_2\rangle \in \mathcal{S}_1 \otimes \mathcal{S}_2$ represent the spin-singlet state of two particles.
- Note that the perfect correlations between them allows us to put $H_1 = H_2 = H$ and to identify $\vec{S}_1$ and $\vec{S}_2$ as follows: Let $s_1(\theta)$ and $s_2(\theta)$ represent the spin states for particles 1 and 2 respectively
- then for an arbitrary angle θ there exists a unit vector $\vec{n}(\theta)$ such that $\vec{S}_1.\vec{n}(\theta)(s_1(\theta)) = \pm s_1(\theta)$ if and only if $\vec{S}_2.\vec{n}(\theta)(s_2(\theta)) = \mp s_2(\theta)$. This relationship allows us to identify $s_2(\theta)$ with the orthogonal complement $s_1^-(\theta)$ of $s_1(\theta)$ and to put $\vec{S}_1 = \vec{S}_2$.

Hence,

$$[S_i(q_1), S_j(q_2)]s = [S_i(q_1), S_j(q_1)]s \quad (3)$$
$$= in\epsilon_{ijk}S_k(q_1)s. \quad (4)$$

However $[S_i(q_1), S_i(q_2)] = 0$ (same i). Consequentally, entanglement in the form of singlet states implies that spin operators can either commute or anticommute beyond the light cone, depending on choice of i and j. This was overlooked by Pauli.

Moreover, since $S_i(q_1)S_j(q_2) \neq 0$ and $S_i(q_1)S_j(q_2) + S_j(q_2)S_i(q_1) = 0$ for $i \neq j$, is only valid for singlet states, it follows that bosons can never be fermions and fermions can never be bosons.

Quantum Gravity and Entanglement

- The first thing to note is that the singlet state is SL(2,C) invariant as is the metric of special relativity. Consequently, singlet states are Lorentz invariant
- Secondly, when extended to general relativity by using the *vierbien* method, the Lorentz in-variance is maintained locally for all singlet states.
- Thirdly, it is important to realize that paired entanglement is a characteristic of all particles with spin regardless of the fields in which they are imbedded.

Quantum Gravity and Entanglement

- Specifically if we have a linearized metric of the form $\tilde{ds}\xi = ds\xi$, where $\tilde{ds} = \gamma^i dx_i$ where γ^i obey the Dirac algebra and ds is a differential eigenvalue of ξ.
- For a singlet state we have $(\tilde{ds}_1, \tilde{ds}_2)(\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1) = ds_1 ds_2(\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1)$
- Duality gives the wave equation

$$(\tilde{\partial}_s, \tilde{\partial}_s)(\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1) = \frac{\partial \psi_1}{\partial s} \otimes \frac{\partial \psi_2}{\partial s} - \frac{\partial \psi_2}{\partial s} \otimes \frac{\partial \psi_1}{\partial s}$$

where s is some form of universal parameter such that for the proper times s_1 and s_2, $s_1 = s_1(s)$ and $s_2 = s_2(s)$.

Conclusion

1. Fermi-Dirac statistics requires pairwise entanglement.
2. It is the presence of singlet states that underlies the Pauli exclusion principle.
3. Bose-Einstein statistics occurs when the entanglement is broken and particles and the spin value of each state are independent of the other.
4. A necessary and sufficient condition for Fermi-Dirac statistics is SL(n,C) invariance.
5. Micro-causality as stated by Pauli does not apply to entangled states or operators. However, causality still applies.
6. Spin entanglement can be incorporated into a gravitational field as an independent axiom.
7. Thank You.