Majorana mass term and CAR algebra of creation and annihilation operators of Dirac and Majorana spinors

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Charge conjugation and Majorana spinors

Dirac equation $\gamma^{\mu}(i\partial_{\mu} - qA_{\mu})\,\psi = m\psi.$

Complex conjugated equation :

$$(\gamma^{\mu})^{*}(-i\partial_{\mu} - qA_{\mu})\psi^{*} = m\psi^{*}.$$

Multiply by $\eta_1\gamma^2$ =>

$$\gamma_D^{\mu}(i\partial_{\mu} + qA_{\mu})\,\eta_1\gamma^2\psi^{*} = m\,\eta_1\gamma^2\psi^{*}.$$

A charge conjugated spinor $\psi^{c} = \eta_1\gamma^2\psi^{*}.$

A charge conjugation operator $(\cdot)^{c} :\ q \to -q.$

Majorana spinor $\psi_M = (\psi_M)^{c} = \frac{1}{2}(\psi + \psi^{c})$ $\quad => q = 0.$

Components of the Majorana spinor

Majorana spinor $\psi_M = (\psi_M)^c = \frac{1}{2}(\psi + \psi^c)$

$=> q = 0.$

$$\psi_L = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

$$\psi_{LM} = \frac{1}{2}(\psi_L + \eta_1 \gamma^2 \psi_L^*) = \frac{1}{2}\begin{pmatrix} \phi \\ -\eta_1 \sigma_2 \phi^* \end{pmatrix}.$$

$$\psi_{RM} = \frac{1}{2}(\psi_R + \eta_1 \gamma^2 \psi_R^*) = \frac{1}{2}\begin{pmatrix} \phi' \\ -\eta_1 \sigma_2 \phi'^* \end{pmatrix} - \text{ the same!}$$

There are only two independent components.

Why Majorana spinors are so important?

- The seesaw mechanism is the most popular model used to explain **very small neutrino masses** compared to the masses of charged leptons.
- Neutrino masses range from fractions of an eV to units of eV, while the masses of charged leptons lie in the range of 0.5-1800 MeV, which is **6-9 orders of magnitude larger**.
- The seesaw mechanism is compatible with the Standard Model.
- The seesaw mechanism predicts existence of the pairs of the very light (observable) and very heavy neutrino.
- Resulting masses of these neutrino are **Majorana masses**.

The seesaw mechanism

Dirac left neutrino ψ_L, sterile neutrino ψ_s.

Dirac mass term + Majorana mass term

$$\overline{(\psi_{sR})} m_D \psi_L + \overline{\psi_s} m_N \psi_s^c + h.c.$$

Mass matrix $m = \begin{pmatrix} 0 & m_D \\ m_D & m_N \end{pmatrix}$, two eigenvalues

$$m_\pm = \frac{m_N \pm \sqrt{(m_N)^2 + 4(m_D)^2}}{2}; \; m_- = -m_\nu, \, m_+ \approx m_N; \quad m_\nu m_+ = (m_D)^2.$$

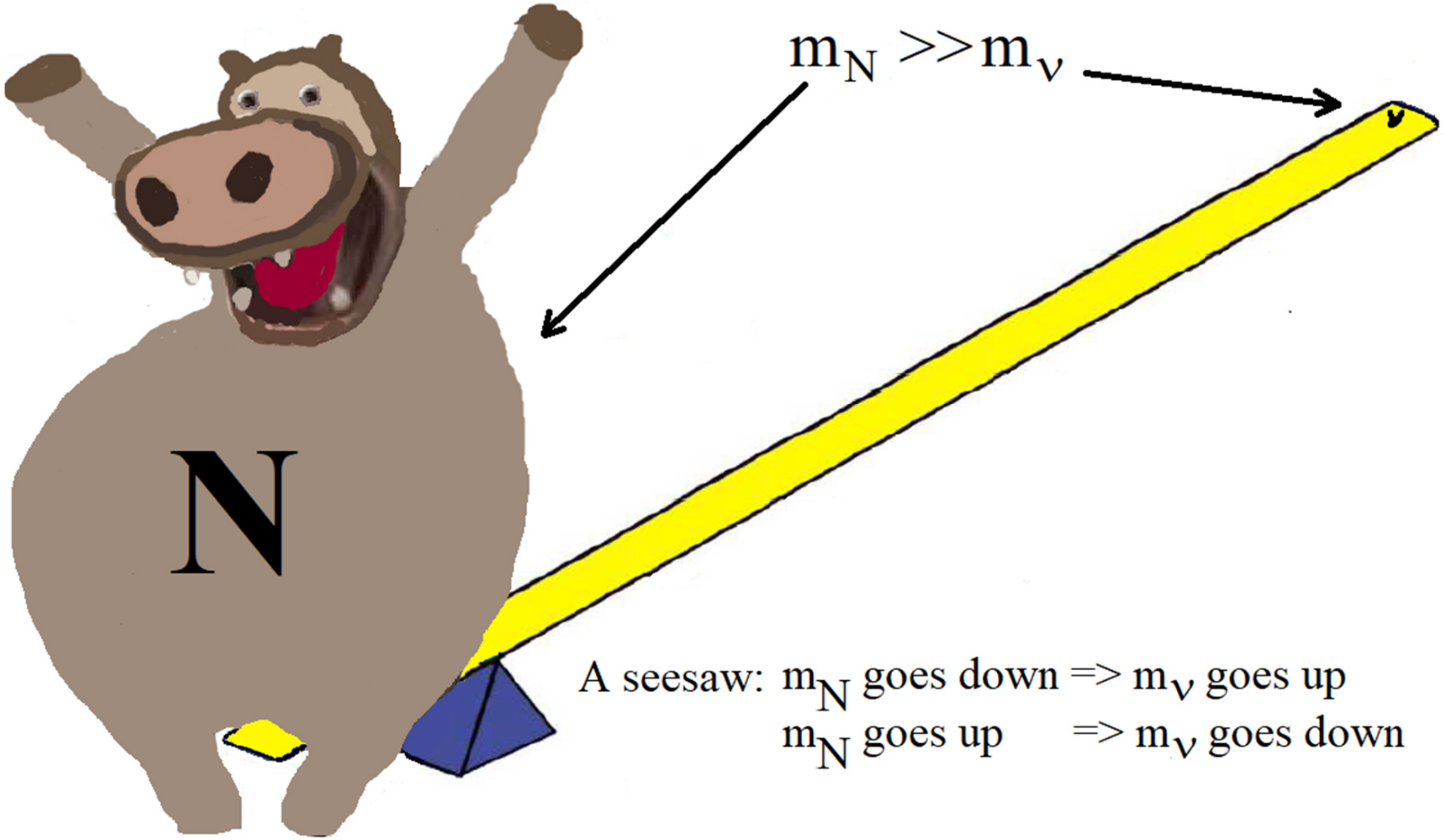
$m_\nu m_N \approx (m_D)^2$ => a seesaw :

m_N goes up => m_ν goes down, m_N goes down => m_ν goes up.

Matrix m can be diagonalized => $\overline{\nu}\, m_\nu\, \nu + \overline{N}\, m_+\, N + h.c.$

=> Majorana neutrinos ν and N.

The seesaw mechanism



A seesaw: m_N goes down => m_ν goes up
m_N goes up => m_ν goes down

Majorana mass term of the Lagrangian – c-theory

$$L_m = -m\bar{\psi}\psi = -m\psi^+\gamma^0\psi = L_m{}^+.$$

Majorana spinor in the Dirac representation :

$$L_{mW} = -m\Big((\phi_1{}'^T\phi_2 - \phi_2^T\phi_1{}') - ((\phi_1{}'^T\phi_2 - \phi_2^T\phi_1{}'))^*\Big),$$

$\phi_1{}' = i\eta_1{}^{-1}\phi_1$; η_1 is a phase factor.

c - theory (without second quantization) :

$$\phi_1{}^+ = \phi_1{}^*,\ \phi_2{}^+ = \phi_2{}^*, \phi_1{}'^T = \phi_1{}',\ \phi_2^T = \phi_2$$

$$=> L_m = 0$$

(Mannheim, P. D. Introduction to Majorana masses.

Int. J. of Theor. Phys. 1984, 23, 643674.)

Majorana representation of the gamma matrices: the Dirac equation at p=0

$$\gamma_{MD}^{0} = -\gamma^{2}\gamma_{D}^{0} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}; \;\; \psi_M = \psi_M^{*} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Solutions of the Dirac equation at $p = 0$

$$\gamma_{MD}^{0} i\partial_0 \, \psi_M = m\psi_M,$$

$$\ddot{\psi}_1 = -m^2\psi_1, \qquad \psi_1(t) = a_1 \cos(mt + \varphi_1),$$

$$\ddot{\psi}_2 = -m^2\psi_2, \quad => \quad \psi_2(t) = a_2 \cos(mt + \varphi_2),$$

$$\psi_3 = \frac{1}{m}\partial_0\psi_2, \qquad \psi_3(t) = -a_2 \sin(mt + \varphi_2),$$

$$\psi_4 = -\frac{1}{m}\partial_0\psi_1, \qquad \psi_4(t) = a_1 \sin(mt + \varphi_1).$$

Majorana mass term of the Lagrangian – q-theory

$$
\begin{aligned}
L_m &= -m\psi_M{}^{+}\gamma^0_{MD}\psi_M = \\
&= -i\, m\,(-a_1{}^{+}a_1\cos(mt+\varphi_1)\sin(mt+\varphi_1) - \\
&\quad - a_2{}^{+}a_2\cos(mt+\varphi_2)\sin(mt+\varphi_2) + \\
&\quad + a_2{}^{+}a_2\cos(mt+\varphi_2)\sin(mt+\varphi_2) + \\
&\quad + a_1{}^{+}a_1\sin(mt+\varphi_1)\cos(mt+\varphi_1)) = 0.
\end{aligned}
$$

There are no assumptions about the operators

$a_1,\ a_2,\ a_1{}^{+}, a_2{}^{+}$ and the quantization procedure.

Only the definition of the Majorana spinor!

The field operator of the Majorana spinor, *p*=0

$$\psi_{DM} = \left(\eta_2 \frac{1-\gamma^2}{\sqrt{2}} \right)^{-1} \psi_M$$

$$\psi_{DM} = \frac{\eta_2^{-1}}{\sqrt{2}} \begin{pmatrix} a_1 \cos(mt+\varphi_1) - ia_1 \sin(mt+\varphi_1) \\ a_2 \cos(mt+\varphi_2) - ia_2 \sin(mt+\varphi_2) \\ -a_2 \sin(mt+\varphi_2) + ia_2 \cos(mt+\varphi_2) \\ a_1 \sin(mt+\varphi_1) - ia_1 \cos(mt+\varphi_1) \end{pmatrix} =$$

$$= \frac{\eta_2^{-1}}{\sqrt{2}} \begin{pmatrix} a_1 \exp(-i(mt+\varphi_1)) \\ a_2 \exp(-i(mt+\varphi_2)) \\ ia_2 \exp(i(mt+\varphi_2)) \\ -ia_1 \exp(i(mt+\varphi_1)) \end{pmatrix}$$

There are no assumptions about the operators a_1, a_2, a_1^+, a_2^+ and the quantization procedure. Only the definition of the Majorana spinor!

The field operator of the Majorana spinor, $p \neq 0$

Normalization factor $n(p) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}}$.

$$\begin{aligned}\psi_{DM} = \eta_2^{-1}\, n(p)\, \big(\;& a_1(p)\, u_1 \exp(-i(p_\mu x^\mu + \varphi_1)) + \\ & + a_2(p)\, u_2 \exp(-i(p_\mu x^\mu + \varphi_2)) + \\ & + i a_2(p)\, v_1 \exp(\, i\,(p_\mu x^\mu + \varphi_2)) - \\ & - i a_1(p)\, v_2 \exp(\, i\,(p_\mu x^\mu + \varphi_1)) \;\big).\end{aligned}$$

u_1, u_2, v_1, v_2 are usual spinor columns in the Dirac representation.

The Hamiltonian of the Majorana spinor, *p*=0

The Lagrangian

$$L = \frac{1}{2}\overline{\psi}_M \gamma^\mu i\partial_\mu \psi_M + \frac{1}{2}(\overline{\psi}_M \gamma^\mu i\partial_\mu \psi_M)^+ - m\overline{\psi}_M \psi_M$$

Corresponding Hamiltonian is

$$H = \frac{\partial L}{\partial \dot{\psi}_M}\dot{\psi}_M - L = \frac{\partial L}{\partial \dot{\psi}_M}\dot{\psi}_M = \overline{\psi}_M \gamma^0 i\partial_0 \psi_M = \psi_M^+ i\partial_0 \psi_M$$

$$p = 0 \Rightarrow H = 0$$

=> Majorana spinor must be massless.

Conclusions

- We have proved that the **Majorana mass term vanishes** not only in the c-theory, which was known, but also in the q-theory (the theory of second quantization).
- We have derived formulas for Majorana spinor field operators without any assumptions about second quantization procedure.
- We have proved that the Hamiltonian of the Majorana spinor at zero momentum is zero. Hence, it is massless.
- The fact that the Majorana mass term vanishes **requires a revision of ideas about the generation of neutrino mass using the seesaw mechanism.**