physical sciences forum

MDPI

Proceedings

Exploring p-Form Mimetic Gravity through $(p+1)$-Brane Fluid Models †

Ilin R. V.

Department of High Energy and Elementary Particle Physics, Saint Petersburg State University, Saint Petersburg 199034, Russia; rireilin@gmail.com

† Presented at the 2nd Electronic Conference on Universe, 16 February–2 March 2023; Available online: https://ecu2023.sciforum.net/.

Abstract: It is well-known that the original mimetic gravity model is nothing more than just pressureless dust that consists of potentially moving particles. This property is an important centerpiece of the theory, which allows it to effectively describe dark matter as a gravitational effect. In this paper, we considering the p-form generalization of this idea and show that they have similar properties to the original formulation. Namely, the matter described by p-form mimetic gravity behaves like fluid of parallel p + 1-dimensional classical branes (or usual bosonic strings for p = 1) that, in general, obey restrictions on their movement. We present alternative formulations for p-form mimetic gravity via dual fields and discuss their cosmological implications.

Keywords: mimetic gravity; p-form electrodynamics; branes; cosmology

Citation: V, I.R. Exploring p-Form Mimetic Gravity through (p + 1)-Brane Fluid Models. *Phys. Sci. Forum* **2023**, 3, x. https://doi.org/10.3390/xxxxx

Published date: 15 February 2023

1. Introduction

In the last decade, much attention has been brought to the problem of describing dark matter. Though the problem is quite old, and now there are plenty of theories, that are trying to deal with it, the dark matter puzzle is still unresolved. Most of the dark matter models suggest adding new fields to the Standard Model to describe new degrees of freedom, and this approach remains dominant even now. However, as there is no clear evidence of the dark matter particles in the current observations, some alternative approaches started to gain popularity. One such alternative is Modified Newtonian Dynamics (MOND) proposed by Mordehai Milgrom [1] in 1983. This theory described the dark matter as a purely gravitational phenomenon, and thus there was no need for any additional matter. However, after the study of the gravitational lensing in the Bullet Cluster, it was clear, that MOND has severe problems in describing dark matter density in the galaxies (see, for example, [2]).

However, there are many other theories that are trying to achieve the same things without having such problems. In the present paper, we consider theories of such kind—mimetic gravity and its generalizations. The original idea was first proposed [3] by V. Mukhanov and A. Chamseddine in 2014. They have shown, that by performing Weyl transformation of the special kind in the Einstein-Hilbert action of General Relativity:

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi. \tag{1}$$

where $\bar{g}_{\mu\nu}$ is auxiliary metric, and φ is new scalar. It is clear, that this formula is invariant under any Weyl transformation of the auxiliary metric, and therefore $\bar{g}_{\mu\nu}$ depends only on 9 independent components. This gauge symmetry also leads to the following constraint on the scalar:

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 1. \quad (2)$$

Mukhanov and Chamseddine have shown, that the resulting theory is just pressureless dust with $\partial_\mu\varphi$ playing the role of 4-velocity, as this relation describes. As there are no additional couplings except for those arising from the coupling with the physical metric $g_{\mu\nu}$, the presence of this matter can be observed as the purely gravitational effect. Such matter can easily mimic dark matter with greater success than MOND as its movement is governed by its own equations of motion and does not require SM fields to exist. In recent studies, many successful modifications of this idea were proposed (see review [4]), and at the moment the development in this direction continues.

Here we will consider a peculiar family of modifications of mimetic gravity, which are using p-forms A instead of the scalar φ. This model was initially considered in [5], but only the cosmology in case $p = 1$ and its Yang-Mills variant [6] was thoroughly analyzed. In this short paper, we will discuss the connection of p-form mimetic gravity with theories of $(p + 1)$-brane fluids – fluids that consist of $(p + 1)$-dimensional classical parallel branes. It appears, as it will be shown in Section 2, that in many cases the motion of the additional mimetic matter in these theories is effectively the motion of $(p + 1)$-brane fluids with some constrains imposed. However, as it will be shown in Section 3.1, for some values of p there are also solutions of p-form mimetic gravity that behave differently. It will be shown, that in four-dimensional spacetime $n = 4$ such non-brane-fluid solutions only appear in the case $p = 1$, whereas for the other values of p mimetic matter behaves exactly like fluid consisting either of usual particles or 2-branes. It should be noted, that the motion of branes in these theories is constrained (for example, in the case of the usual mimetic gravity, the potentiality condition $u_\mu = \partial_\mu\varphi$ on 4-velocity). Finally, in Section 3.2. we will briefly discuss the cosmological application of the methods and propose the new FLRW-solution for the case $p = 2$.

2. Methods

The p-form mimetic gravity can be formulated by construction described in [5]. The key ingredient is the generalization of the Weyl transformation written in (1):

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\left(\beta\langle F, F\rangle_{\bar{g}}\right)^{\frac{1}{p+1}}, \quad (3)$$

where we used notation:

$$\langle F, F\rangle_{\bar{g}} \equiv \frac{1}{(p+1)!}\bar{g}^{\alpha_1\beta_1} \dots \bar{g}^{\alpha_{p+1}\beta_{p+1}} F_{\alpha_1\dots\alpha_{p+1}} F_{\beta_1\dots\beta_{p+1}}. \quad (4)$$

Here $F_{\alpha_1\dots\alpha_{p+1}}$ is fully-antisymmetric field strength of the p-form potential $A_{\alpha_1\dots\alpha_p}$. For brevity, we will use standard differential form notation for form differential d, wedge product $\wedge$, and the forms themselves, in which this relation is simply $F = dA$. The parameter β is equal to ± 1, and we also require the quantity in the brackets to be positive. In the particular case p = 0 this formula coincides with the (1).

To obtain the corresponding mimetic theory one has to substitute this formula into the gravitational action and then treat p-form potential A and auxiliary metric $\bar{g}_{\mu\nu}$ as the new arguments of the action. As it was shown in [5], there is a much simpler equivalent formulation for this theory, similar to the one obtained in [7] for the case $p = 0$. To write it, one needs to use the generalization of condition (2), which arises for (3) from the gauge invariance under Weyl transformations of the metric $\bar{g}_{\mu\nu}$:

$$g^{\alpha_1\beta_1} \dots g^{\alpha_{p+1}\beta_{p+1}} F_{\alpha_1\dots\alpha_{p+1}} F_{\beta_1\dots\beta_{p+1}} = \beta. \quad (5)$$

Now one can introduce Lagrange multiplier λ and place this constraint in front of it. The resulting action in the form notation will take the form:

$$S[g_{\mu\nu}, A, \lambda] = S_{EH} + \frac{1}{2}\int \lambda\,(\sigma F \wedge\star F - \beta \star 1), \quad (6)$$

where S_{EH} is Einstein-Hilbert action, and $\sigma = (-1)^n$ is the sign of metric determinant (we are working with space-time signature $+ - \cdots -$). The main interest here is presented by the equations of motion, that are obtained from variations with respect to λ and A:

$$d(\lambda \star F) = 0, \tag{7}$$

$$\langle F, F \rangle_g = \beta, \tag{8}$$

As was stated in the introduction, this theory and the theory of fluid of $(p+1)$-dimensional branes are connected to each other. This connection is subtle, and to clearly see it we propose using $(q+1)$-form field strengths $B \equiv \lambda \star F$ instead of the original F (where $q \equiv n - p - 2$). From equation (7) it immediately follows, that B is indeed the field strength with some q-form potential W. But the main benefit comes from expressing the equation (8) in terms of B:

$$\lambda = \pm\sqrt{\beta\sigma\langle B, B \rangle_g}. \tag{9}$$

Thus, we got rid of the Lagrange multiplier, and now the theory can be completely described in terms of the physical metric $g_{\mu\nu}$ and the "dual" potential W. However, as the field strength F obeys Bianchi identity, it imposes a constraint on the dual field strength:

$$dF \equiv d\left(\frac{1}{\sqrt{\beta\sigma\langle B, B \rangle_g}} \star B\right) = 0. \tag{10}$$

This constraint will be the only restriction on B apart from the Einstein's equations. One may find the new action for the theory in terms of the dual field strength and the corresponding q-form potential:

$$S[g_{\mu\nu}, W] = S_{EH} + \int \sqrt{\beta\sigma\langle B, B \rangle_g}. \tag{11}$$

Now we want to turn to the brane fluid models. As it was stated earlier, these models describe fluids, consisting of parallel branes. We will assume, that the $(p+1)$-dimensional parallel branes M_h are level surfaces for some functions h^A $(A = 1, 2, \dots q+1)$. We then introduce the coordinates $\hat{x}^i$ on each brane in such a way, that $x^\mu(\hat{x}^i)$ is a smooth embedding of M_h into spacetime. Then, the simplest action for this stack of branes is simply the sum of the actions for each:

$$S_{BF}[x^\mu] = \int dh^A \int_{M_h} \sqrt{\hat{g}}\, d\hat{x}. \tag{12}$$

Note, that one may choose quantities $h^A, \hat{x}^i$ as coordinates of the spacetime. It was shown [8], that this theory can be equivalently rewritten by performing diffeomorphism that maps these coordinates to the arbitrary ones x^μ. Here we will not discuss the detail and just restrict ourselves to using the resulting action:

$$S_{BF}\left[g_{\mu\nu}, h^A\right] = \int dx \sqrt{\omega[h^A]}, \tag{13}$$

where $\omega \equiv \sigma \det(\omega^{AB})$, and the matrix ω^{AB} is defined as follows:

$$\omega^{AB} \equiv g^{\mu\nu}\partial_\mu h^A \partial_\nu h^B. \tag{14}$$

It should be stressed, that unlike model (12) this equivalent formulation treat brane level functions h^A as the argument of the action, not the coordinate of the spacetime.

There is one last step, that connects (13) with the dual formulation of p-form mimetic gravity (11). Note, that by definition, one can rewrite determinant ω by using Levi-Civita tensor. After some simplifications, the result may be presented as follows:

$$\omega = \langle dC, dC \rangle_g, \tag{15}$$

where:

$$C[h^A] \equiv \frac{1}{(q+1)!}\varepsilon_{A_1\dots A_{q+1}} h^{A_1} dh^{A_2} \wedge \dots \wedge dh^{A_{q+1}}. \quad (16)$$

By comparing (15) and the integrand in (11), one finally arrives at the destination point: the theory that describes fluid consisting of the $(p+1)$-dimensional parallel branes can be obtained from the "dual" p-form mimetic gravity action by performing the transformation in action (11) of the form $W \to C[h^A]$. The analysis of this fact is presented in the next section.

3. Results and Discussion

3.1. Particular Cases

The obtained result has many consequences regarding the solutions in both theories. It is instructive to turn to another excellent example of theory, which is obtained by the transformation in action—the original mimetic gravity itself. Indeed, recall that it was also obtained by performing field transformation (1) in the Einstein-Hilbert action. In [7,9] it was argued, that the presence of the scalar derivatives alone in this transformation gives rise to the new mimetic degrees of freedom in theory. Moreover, from the equation of motion of mimetic gravity

$$G_{\mu\nu} + G\partial_\mu\varphi\partial_\nu\varphi = 0, \quad (17)$$

it is clear, that the solutions of the General Relativity are also preserved by the transformation. Thus, one may expect the same behavior in the case of a completely different transformation $W \to C[h^A]$, that connects brane fluid model (13) and p-form mimetic gravity (11). Namely, it is expected, that the solutions of p-form mimetic gravity will be preserved by the field transformation, and also that new degrees of freedom will arise after the transformation, i.e. in theory (13).

In case $p = 0$, which is simply original mimetic gravity (1), it is obviously true. Indeed, it is already known, that this theory describes pressureless dust, which consists of parallel particles (1-branes) moving alongside geodesics. In other words, all the solutions of the $p = 0$ mimetic gravity are preserved after the transformation in a new theory of 1-brane fluid. It is also clear, that 1-brane fluid theory contains extra degrees of freedom compared to the 0-form mimetic model, as the latter describes only potentially moving dust, while the former doesn't restrict particles' movement.

Another interesting case is $p = n - 2$, which corresponds to the branes of co-dimension 1. This theory also behaves as expected, but there is more. In fact, for this p the transformation $W \to C[h^A]$ doesn't bring any new degrees of freedom. The reason is quite trivial: the quantity q is equal to $n - p - 2 = 0$, so from (16) it follows, that $C[h] = h$. Therefore, for $p = n - 2$ the corresponding $n - 2$-form mimetic gravity is *exactly* the (n-1)-brane fluid.

However, the other values of p doesn't met the abovementioned expectations. The reason behind this fact lies in the transformation formula. As the simplest example, we consider the most interesting case $n = 4$. In this case, there are three allowed values for p = 0, 1, 2, in total. The cases p = 0 and 2 were already considered above, and the case $p = 1$ is the only nontrivial one. The transformation formula has the form:

$$C[h^A] = \frac{1}{2}\varepsilon_{AB} h^A dh^B. \quad (18)$$

This formula differs from the abovementioned transformations. Without going into much detail, it can be shown, that the transformation (16) is actually just the Clebsch [10] representation of the *arbitrary* $n - 2$-form C (modulo gauge transformations $C \to C + d\alpha$, which doesn't change anything, as C enters (15) and hence (13) only through its differential). For $p = n - 2$ the transformation (16) is plain trivial. The common thread between two cases is that the arbitrary value of C can be transformed into h^A as it is prescribed by (16). However, from the Formula (18) it is clear, that not every 1-form C can be presented this way. It can be easily shown further, that at the level of equations of motion, not every solution of the equations of motion in 1-form mimetic gravity behaves like 2-brane (string) fluid. However, for those that do behave like that, the motion will be constrained by the generalized potentiality conditions, as it was in case $p = 0$.

3.2. Cosmology

As the main purpose of mimetic gravity is its cosmological applications, we will briefly discuss the equation of state for the obtained models. We will once again restrict ourselves to the case $n = 4$ and also will use the value $\beta = -1$ as it corresponds to the branes being timelike. For $p = 0$ we already know the answer – the dust has the standard equations of motion of the perfect fluid with the pressure to density ratio $w = 0$. The other two cases obviously don't have any FLRW-solutions, because both parallel strings and branes are breaking isotropy. Therefore, instead of discussing these theories, we will describe, what happens in their closest generalizations. Namely, instead of using the single form A in (3), one can use the triplet of forms A^a (a = 1,2,3) and require the action to be SO(3)-invariant. In that case, the formulae (5)-(11) are still valid, but the connection to the brane fluid theory (16), in general, is lost. However, if the spacetime has FLRW-symmetry, one may still connect the solutions of SO(3)-invariant version of (11) to the solutions of the 3 copies of p-form fluid theory (16). The reason for that is the isotropy requirement: in SO(3)-invariant mimetic p-form all three contributions under the root in (11) are equal, and hence $\sqrt{\beta\sigma\langle B^a, B^a\rangle_g} = 3\sqrt{\beta\sigma\langle B, B\rangle_g}$, where B are field strenth, built from only one component of the triplet A^a.

The cosmology for the case $p = 1$ was studied in [5]. The resulting equation of state for the mimetic matter was $w = -1/3$. This answer naturally arises from the simple cosmology of the universe filled with cosmic string gas. It is interesting to look at the values of electric E^a and magnetic H^a components of the solution, presented in the mentioned work. The former is non-zero, and the latter is zero. It can be shown, that it means, that the conditions $F \wedge F = 0$ and hence $B \wedge B = 0$ are satisfied in this case. From the geometrical point of view, it means, that at each point B is proportional to some volume form of 2-surface. It is straightforward to show, that by using Bianchi identity $dB = 0$ it means, that locally B can be presented as $d\alpha \wedge d\gamma$ for some γ, β, and therefore, its corresponding potential can be presented in the form (18). Thus, the FLRW-solution obtained in [5] is precisely the string-fluid solution for each of the three components of the triplet.

The same logic can be applied to the last nontrivial case $p = 2$. In this case, the components of triplets transforms as scalars W^a with respect to diffeomorphisms, which simplifies the calculations. As $\beta = -1$, the gradient of W^a must be spacelike in order to keep radicand positive in the denominator in the Formula (10). For the spatially flat universe, it is easy to check, that the solution of the equations of motion (10) is

$$W^a = Y\delta^a_\mu x^\mu, \tag{19}$$

where $Y =$ const, and δ^a_μ is Kronecker symbol. The corresponding pressure to density ratio $w = -2/3$. This completely agrees with the above reasoning and also previous calculations of the cosmology in universe filled with the gas of domain walls [11].

4. Conclusions

In the present paper, we have shown, that the p-form mimetic gravity model (3) can be connected with the model that describes fluid consisting of (p+1)-dimensional parallel branes. We have used dual field strength to solve one of the equations of motion of the original p-form mimetic gravity with respect to the Lagrange multiplier and rewrite the theory only in terms of the dual fields. The action, which leads to these equations directly by variational principle is also presented by the Formula (11). For that new action, we have shown, that by applying differential field transformation to it one can obtain $(p+1)$-brane fluid model. We discussed different particular cases of this connection and showed, that in cases $p = 0$ and $p = n - 2$ all the mimetic gravity solutions behave like fluids that consist of particles and co-dimension 1-branes respectively. We also discussed the other cases by considering the example $p = 1, n = 4$ and showing, that the field transformation that is supposed to lead to the brane fluid formulation, is significantly different from that in the previous cases. Finally, we discussed the cosmology in 4-dimensional setting and proposed a new FLRW-solution for the case $p = 2$ with properties of domain wall gas.

At the end, the one last point should be mentioned. The proposed method can be applied not only to the "pure" p-form mimetic gravity, but also to the situations, where there are multiple terms in the brackets in (6). This fact was actually used when we discussed FLRW-solutions for SO(3)-generalization of the cases $p = 1, 2$. However, one may also combine different types of mimetic theories (with different p) to realize complex cosmological scenarios. This topic is very rich and requires thorough investigation in the future.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author is grateful to S.A. Paston for useful discussions and commentaries on the topic.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Milgrom, M. A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. *Astrophys. J.* **1983**, *270*, 365–370. https://doi.org/10.1086/161130.
2. Scott, D.; White, M.; Cohn, J.D.; Pierpaoli, E. Cosmological Difficulties with Modified Newtonian Dynamics (or: La Fin du MOND?). *arXiv* **2001**, arXiv:astro-ph/0104435.
3. Chamseddine, A.H.; Mukhanov, V. Mimetic dark matter. *J. High Energ. Phys.* **2013**, 135. https://doi.org/10.1007/JHEP11(2013)135.
4. Sebastiani, L.; Vagnozzi, S.; Myrzakulov, R. Mimetic gravity: A review of recent developments and applications to cosmology and astrophysics. *Adv. High Energy Phys.* **2017**, *2017*, 3156915. https://doi.org/10.1155/2017/3156915.
5. Gorji, M.A.; Mukohyama, S.; Firouzjahi, H.; Mansoori, S.A.H. Gauge mimetic cosmology. *JCAP* **2018**, *2018*, 047. https://doi.org/10.1088/1475-7516/2018/08/047.
6. Gorji, M.A.; Mukohyama, S. Firouzjahi. Cosmology in Mimetic SU(2) Gauge Theory. *JCAP* **2019**, *2019*, 018. https://doi.org/10.1088/1475-7516/2019/05/019.
7. Golovnev, A. On recently proposed mimetic gravity. *Phys. Lett.* **2014**, *728*, 39. https://doi.org/10.1016/j.physletb.2013.11.026.
8. Sheykin, A.A.; Paston, S.A. Field-theoretical formulation of Regge–Teitelboim gravity. *Phys. At. Nucl.* **2016**, *79*, 1494–1496. https://doi.org/10.1134/S1063778816100124.
9. Sheykin, A.A.; Solovyev, D.P.; Sukhanov, V.V.; Paston, S.A. Modifications of Gravity Via Differential Transformations of Field Variables. *Symmetry* **2020**, *12*, 240. https://doi.org/10.3390/sym12020240.
10. Rund, H.; Forbes, W.F. *Topics in Differential Geometry*, 1st ed.; Elsevier: Amsterdam, The Netherlands, 1975; pp. 111–133.
11. Kolb, E.W.; Turner, M.S. The Early Universe. *Front. Phys.* **1990**, *69*, 1–547. https://doi.org/10.1201/9780429492860.