Harmonic oscillator in the context of the Extended Uncertainty Principle

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Abstract

At large scale distances where the space-time is curved due to gravity, a nonzero minimal uncertainty in the momentum, $(\Delta P)_{\min}$, is emerged. The presence of a minimal uncertainty in momentum allows a modification to the quantum uncertainty principle, that is known as Extended Uncertainty Principle (EUP). In this work, we handle the harmonic oscillator problem in the EUP scenario and obtain analytical exact solutions in classical and semiclassical domains. In the classical context, we establish the equations of motion of the oscillator and show that the EUP-corrected frequency is depending on the energy and deformation parameter. In the semiclassical domain, we derive the energy eigenvalue levels and demonstrate that the energy spectrum depends on n^2, as the feature of hard confinement. Finally, we investigate the impact of the EUP on the harmonic oscillator's thermodynamic properties by using the EUP-corrected partition functions in classical limit in the (A)dS backgrounds.

A brief review of the EUP

In this section, we review the development of the EUP in the AdS and dS, respectively.

AdS background

S. Mignemi, Mod. Phys. Lett. A **25**, 1697 (2009).

$$\left[\hat{X},\hat{P}\right]=i\hbar\left(1+\alpha\hat{X}^{2}\right),$$ where $\alpha=\frac{\alpha_0}{L}$, while α_0 is a numerical constant and L is large distance scale.

$$\Delta X\Delta P\geq\frac{\hbar}{2}\left(1+\alpha\left(\Delta X\right)^{2}\right)$$ leads to an absolute minimal uncertainty in momentum $(\Delta P)_{\text{min}}=\hbar\sqrt{\alpha}.$

By setting modified position and momentum operators $\hat{X}=\hat{x},$ and $\hat{P}=\left(1+\alpha\hat{x}^{2}\right)\hat{p},$

we can certify Eq. (1), while the operators $\hat{x}$ and $\hat{p}$ obey the ordinary Heisenberg algebra $[\hat{x},\hat{p}]=i\hbar.$

these deformed operators are Hermitian under the modified scalar product

$$\langle\psi\,|\phi\rangle=\int_{-\infty}^{+\infty}\frac{dx}{1+\alpha x^{2}}\psi^{*}\left(x\right)\phi\left(x\right).$$

dS background

$$\left[\hat{X}, \hat{P}\right] = i\hbar\left(1 - \alpha\hat{X}^2\right),$$

generates the EUP of the form

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left(1 - \alpha\left(\Delta X\right)^2\right).$$

In this case, a minimal momentum uncertainty value does not emerge.

we realize the modified position and momentum operators as $\hat{X} = x, \quad \text{and} \quad \hat{P} = \frac{\hbar}{i}\left(1 - \alpha x^2\right)\frac{d}{dx}.$

scalar product can be defined with

$$\langle\psi\,|\phi\rangle = \int_{-1/\sqrt{\alpha}}^{+1/\sqrt{\alpha}} \frac{dx}{1 - \alpha x^2}\psi^*\left(x\right)\phi\left(x\right).$$

The modified eigenvalue spectrum of the modified momentum operator, p, can be determined by

AdS background

$$\frac{\hbar}{i}\left(1+\alpha x^{2}\right)\frac{d}{dx}\psi\left(x\right)=p\psi\left(x\right),$$

dS background

$$\frac{\hbar}{i}\left(1-\alpha x^{2}\right)\frac{d}{dx}\psi=p\psi.$$

where $\psi\left(x\right)$ is the momentum eigenfunction

$$\psi\left(x\right)=Ne^{\frac{ip}{\hbar\sqrt{\alpha}}\arctan\left(\sqrt{\alpha}x\right)},$$

can be normalized

$$\psi=\mathcal{N}e^{\frac{ip}{\hbar\sqrt{\alpha}}\operatorname{arctanh}\left(\sqrt{\alpha}x\right)}.$$

cannot be normalized

The classical solution

One can express the classical limit of the EUP scenario by substituting commutators with the Poisson brackets

$$\frac{1}{i\hbar}[A,B] \to \{A,B\}.$$

In the deformed classical mechanics, the general form of the Poisson brackets takes the form of

$$\{A,B\} = \left(\frac{\partial A}{\partial X_i}\frac{\partial B}{\partial P_j} - \frac{\partial A}{\partial P_i}\frac{\partial B}{\partial X_j}\right)\{X_i,P_j\} + \frac{\partial A}{\partial X_i}\frac{\partial B}{\partial X_j}\{X_i,X_j\}.$$

Correspondingly, the classical equations of motion become

$$\dot{X} = \{X,H\} = \{X,P\}\frac{\partial H}{\partial P},$$

$$\dot{P} = \{P,H\} = -\{X,P\}\frac{\partial H}{\partial X},$$

Now, let us handle the harmonic oscillator problem

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2,$$

with the initial condition

$$X\left(t = 0\right) = 0,$$

where m and ω denote the oscillator's mass and angular frequency, respectively.

In one dimensional AdS background the Poisson bracket of the position and momentum operators evolves

$$\{X, P\} = \left(1 + \alpha X^2\right),$$

and the equations of motion read

$$\begin{aligned} \dot{X} &= \left(1 + \alpha X^2\right) \frac{P}{m}, \\ \dot{P} &= -\left(1 + \alpha X^2\right) m\omega^2 X. \end{aligned}$$

We observe that additional terms proportional to α arise in the $\dot{X}$ and $\dot{P}$ equations.

Since Hamiltonian corresponds to the total energy of the system, it should be conserved like in the standard classical mechanics.

$$\frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 = E.$$

In order to check this, we choose

$$z = \arctan\left(\sqrt{\alpha}X\right),$$

to solve the equations of motion. we get

$$\ddot{z} + \omega^2 \frac{\sin(z)}{\cos^3(z)} = 0.$$

By integrating we get

$$\left(\frac{dz}{dt}\right)^2 + \omega^2 \tan^2 z = \frac{2E\alpha}{m}.$$

The latter equation is identical to those of a particle's motion in an effective potential $V(z) = \omega^2 \tan^2 z$.

Integration yields

$$z=-i\arctan\left\{\frac{\sqrt{\frac{2E\alpha}{m}}\left[Ce^{2i\sqrt{\frac{2E\alpha}{m}+\omega^2}t}-1\right]}{\sqrt{\frac{2E\alpha}{m}+2\left(\frac{2E\alpha}{m}+2\omega^2\right)Ce^{2i\sqrt{\frac{2E\alpha}{m}+\omega^2}t}+\frac{2E\alpha}{m}C^2e^{4i\sqrt{\frac{2E\alpha}{m}+\omega^2}t}}}\right\}.$$

Here, C is a constant of integration which will be determined from the initial conditions.

we obtain the solution of the AdS oscillator

$$X(t)=\frac{\sqrt{\frac{2E}{m\omega^2}}\sin\left[\omega\sqrt{1+2E\alpha}t\right]}{\sqrt{1+\frac{2mE\alpha}{\omega^2}\cos^2\left[\omega\sqrt{1+2E\alpha}t\right]}}.$$

$$P(t)=\sqrt{2mE\left(1+\frac{2mE\alpha}{\omega^2}\right)}\frac{\cos\left[\omega\sqrt{1+2E\alpha}t\right]}{\sqrt{1+\frac{2mE\alpha}{\omega^2}\cos^2\left[\omega\sqrt{1+2E\alpha}t\right]}}.$$

It is worth noting that the solutions are still periodic, but the effective frequency depends on the total energy. Also, a correction of order $2\alpha E$ emerges in the amplitudes and they are no longer sinusoidal.

Next, we perform the same calculation for the dS oscillator. In this case, the equations of motion reads

$$\begin{aligned}\dot{X} &= \left(1-\alpha X^2\right)\frac{P}{m},\\ \dot{P} &= -\left(1-\alpha X^2\right)m\omega^2 X^2.\end{aligned}$$

By defining a new variable $y = \operatorname{arctanh}\left(\sqrt{\alpha}X\right)$ we obtain an effective potential $V\left(y\right) = \omega^2 \tanh^2 y.$

Then, the solutions emerge

$$X\left(t\right) = \frac{\sqrt{\frac{2mE}{\omega^2}}\sin\left[\omega\sqrt{1-2E\alpha}t\right]}{\sqrt{1-\frac{2E\alpha}{k^2}\cos^2\left[\omega\sqrt{1-2E\alpha}t\right]}},$$

$$P\left(t\right) = \sqrt{2mE\left(1-\frac{2mE\alpha}{\omega^2}\right)}\frac{\cos\left[\omega\sqrt{1-2\alpha E}t\right]}{\sqrt{1-\frac{2mE\alpha}{\omega^2}\cos^2\left[\omega\sqrt{1-2E\alpha}t\right]}}.$$

In Fig. 1, we plot the temporal behavior of X and P in AdS background.

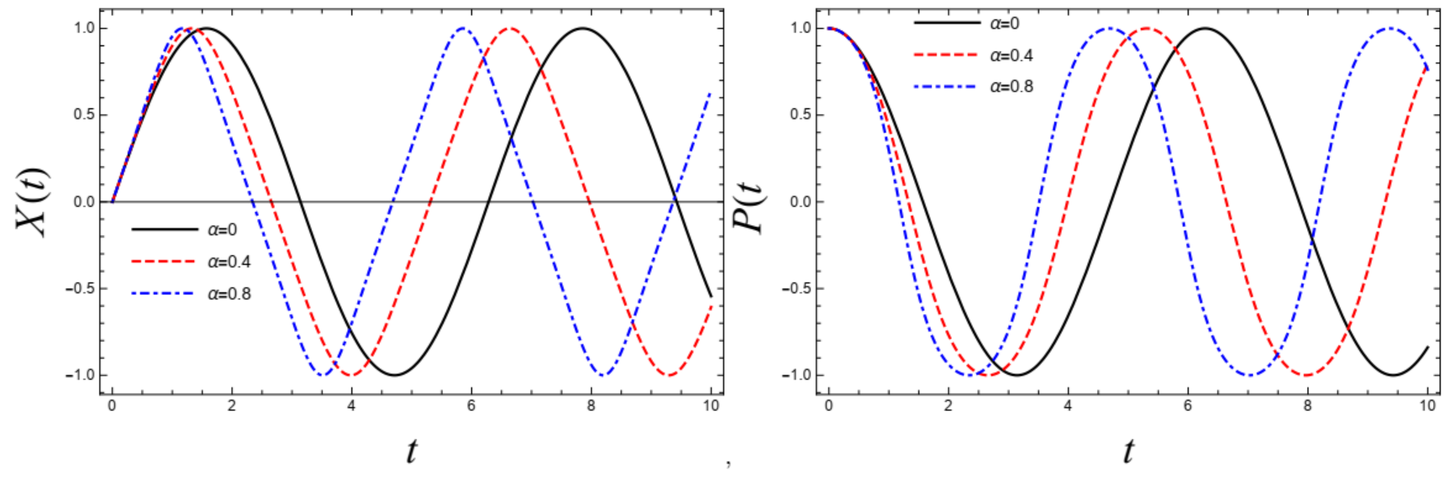

Figure 1: The temporal behavior of X and P in AdS background. We set $m = \omega = 1$, and $E = 1/2$.

In Fig. 2, we plot the temporal behavior of X and P in dS background.

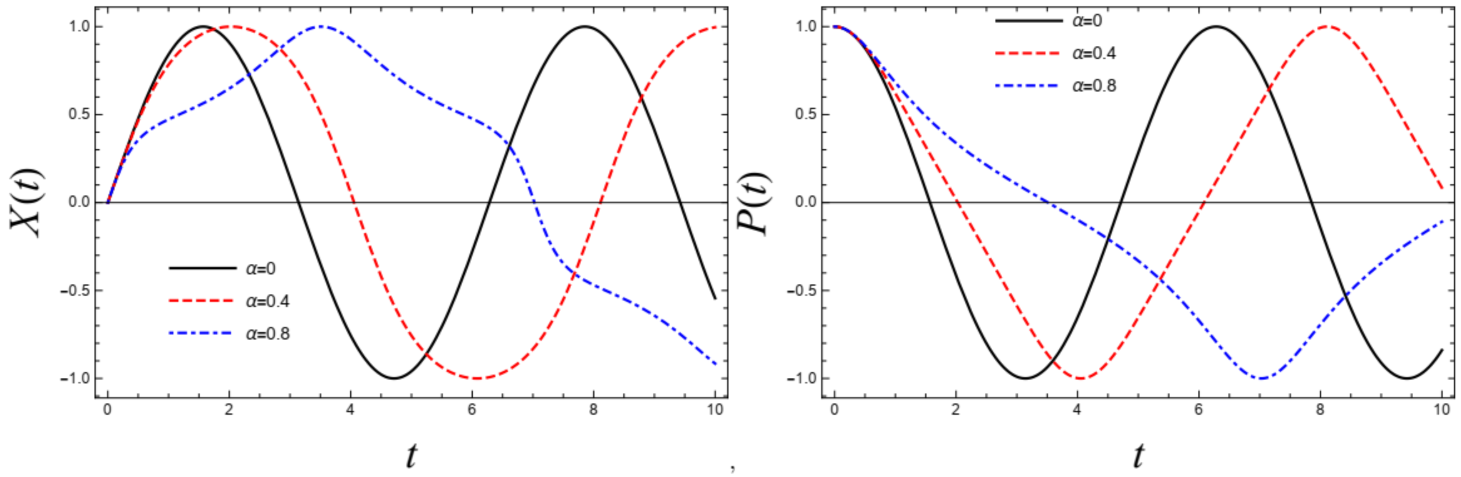

Figure 2: The temporal behavior of X and P in dS background. We set $m = \omega = 1$, and $E = 1/2$.

Semiclassical solution

Before discussing the EUP harmonic oscillator problem in a semi-classical approach, we would like to review briefly the derived results in the context of the quantum mechanics

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 = E,$$

where the constant E represents the total energy. According to the Newtonian symplectic form, $dw = pdx$, the Bohr-Sommerfeld quantization condition reads

$$\oint p\,(x)\,dx = \pi\hbar \left(n + \frac{1}{2}\right),$$

while the notation $\oint$ corresponds to the integral over one complete period of the classical motion. Solving for $p \geq 0$, we get

$$p\,(x) = \sqrt{2mE - m^2\omega^2 x^2},$$

and then we rewrite

$$\int_{-\sqrt{\frac{2E}{m\omega^2}}}^{\sqrt{\frac{2E}{m\omega^2}}} \sqrt{2mE - m^2\omega^2x^2}dx = \frac{4E}{\omega}\int_0^{+1} \sqrt{1-\xi^2}d\xi = \frac{\pi}{4},$$

with $\xi = \sqrt{\frac{m\omega^2}{2E}}x$. By imposing the Bohr-Sommerfeld quantization condition, we obtain the spectrum

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

In AdS background, the measure of the position space is modified with $dx \to \frac{dX}{1+\alpha X^2}$, and the Bohr-Sommerfeld quantization rule can be represented by the formula

$$\oint P(X)\,\frac{dX}{1+\alpha X^2} = \pi\hbar\left(n+\frac{1}{2}\right),$$

where

$$P(X) = \sqrt{2mE - m^2\omega^2 X^2}.$$

Thus, we get

$$\int_{-\sqrt{\frac{2E}{m\omega^2}}}^{\sqrt{\frac{2E}{m\omega^2}}} dX\,\frac{\sqrt{2mE - m^2\omega^2 X^2}}{1+\alpha X^2} = \pi\hbar\left(n+\frac{1}{2}\right).$$

Following a straightforward algebra, we find

$$E_n^{AdS} = \hbar\omega\left(n+\frac{1}{2}\right) + \frac{\hbar^2\alpha}{2m}\left(n+\frac{1}{2}\right)^2.$$

E_n^{AdS} tends to $\hbar\omega\left(n+\frac{1}{2}\right)$ as α goes to zero.

n^2 dependency of eigenvalue function represents the hard confinement.

In dS background, we perform a series of similar algebraic manipulations, and obtain the energy eigenvalues in the form of

$$E_n^{dS} = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{\hbar^2\alpha}{2m}\left(n+\frac{1}{2}\right)^2, \quad n = 0, 1, ..., n_{\max}.$$

In the present case, unlike the previous one, energy eigenvalue function has an upper bound value. To determine the quantum number which corresponds to that, we use

$$\frac{dE_n^{dS}}{dn} = 0,$$

and obtain the maximum value of the quantum number as

$$n_{\max} = \frac{m\omega}{\hbar\alpha} - \frac{1}{2}.$$

We notice that in the absence of deformation parameter, this upper bound goes to infinity as in the ordinary case. we depict a finite number of energy eigenvalue levels versus n in Fig. 3.

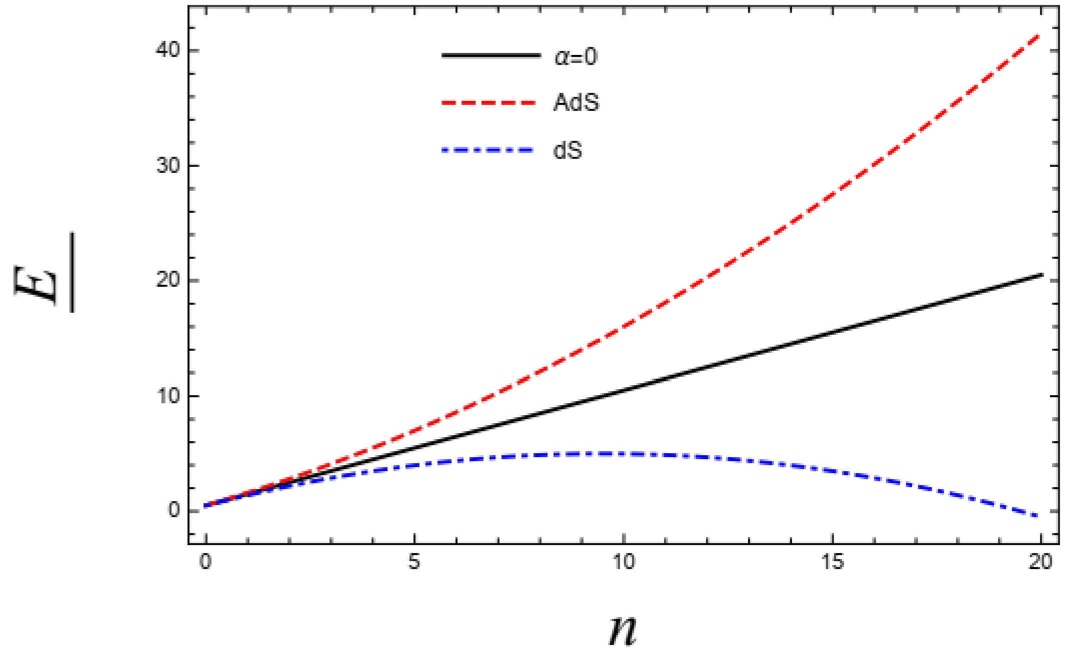

Figure 3: Energy spectrum $\frac{E_n}{\hbar\omega}$ as a function of n.

We observe that the EUP-correction terms decrease the values of the eigenenergy levels in the dS background, but increase them in the AdS domain.

Classical partition function

The canonical partition function in classical approach is based on the following equation

$$Z = \frac{2\pi}{\hbar} \int \int dx dp e^{-\beta H},$$

with $\beta = \frac{1}{K_B T}$. Here, H, K_B and T denote the classical Hamiltonian, the Boltzmann constant, and the thermodynamic temperature of the system, respectively.

Using the deformed commutation relations with the undeformed Hamiltonian,

$$Z = \frac{2\pi}{\hbar} \int \int \frac{dX dP}{J} e^{-\beta H},$$

and the effects of the EUP deformation are taken into consideration via the Jacobian, J, and the phase space elements

$$J = 1 \pm \alpha X^2.$$

in the AdS background

$$Z^{AdS} = \frac{2\pi}{\hbar} \int_{-\infty}^{+\infty} dP e^{-\frac{\beta P^2}{2m}} \int_{-\infty}^{+\infty} \frac{dX}{1+\alpha X^2} e^{-\frac{\beta m \omega^2 X^2}{2}}.$$

Then, we expand the factor $\left(1+\alpha X^2\right)^{-1}$ up to first-order. After performing the integration, we find

$$Z^{AdS} \simeq 4\pi^2 \tau \left[1 - 2\Theta\tau\right],$$

where

$$\tau = \frac{K_B T}{\hbar\omega} = \frac{1}{\beta\hbar\omega}, \quad \text{and} \quad \Theta = \frac{\hbar\alpha}{2m\omega}.$$

In the dS background, the range of permitted values of X is bounded

$$\frac{-1}{\sqrt{\alpha}} \leq X \leq \frac{1}{\sqrt{\alpha}}.$$

Therefore, after expanding the factor $\left(1-\alpha X^2\right)^{-1}$, we perform the integral via this interval.

$$Z^{dS} \simeq 4\pi^2\tau\left[(1+2\Theta\tau)\mathrm{erf}\left(\sqrt{\frac{1}{4\Theta\tau}}\right) - 2\sqrt{\frac{\Theta\tau}{\pi}}e^{-\frac{1}{4\Theta\tau}}\right],$$

where $\mathrm{erf}(x)$ is the Gauss error function. Taking into consideration the asymptotic expansion of $\mathrm{erf}(x)$ function for large x

$$\mathrm{erf}\,(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\left(x^{-1} - \frac{x^{-3}}{2} + ...\right),$$

we rewrite the partition function up to the first order of Θ as follows:

$$Z^{dS} \simeq 4\pi^2\tau\left[1+2\Theta\tau\right].$$

Next, we examine the thermal quantities of the classical harmonic oscillator such as entropy, internal energy and specific heat functions in both backgrounds with the following formulas

$$\begin{aligned} S &= \ln Z + \tau \frac{\partial}{\partial \tau} \ln Z, \\ U &= \tau^2 \frac{\partial}{\partial \tau} \ln Z, \\ C &= 2\tau \frac{\partial}{\partial \tau} \ln Z + \tau^2 \frac{\partial^2}{\partial \tau^2} \ln Z. \end{aligned}$$

Here, we employ only numerical methods for the calculations and depict the thermodynamic functions versus the reduced temperature for $\Theta = 10^{-2}$. At first, we display the entropy function in Fig. 4.

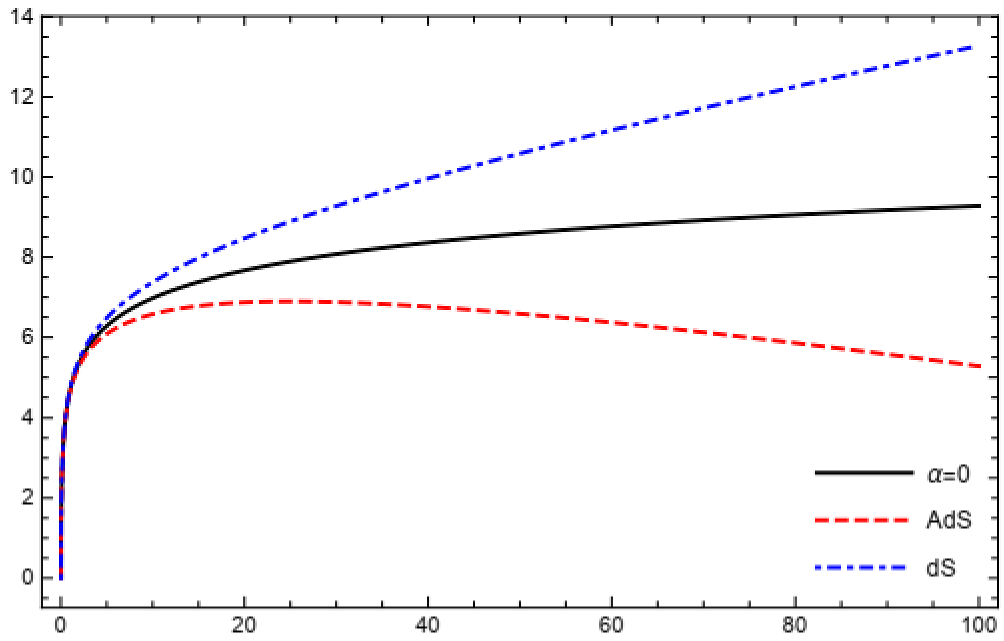

Figure 4: The classical entropy versus τ.

We observe that the in dS background, the entropy increases with the increment of τ. We notice that in this case entropy diverges when the reduced temperature tends to infinity. In the AdS background, at low temperature the entropy rapidly increases as the other case, and then it goes to zero when the reduced temperature approaches to a critical value.

Then, in Fig. 5 we demonstrate the internal energy function's behavior.

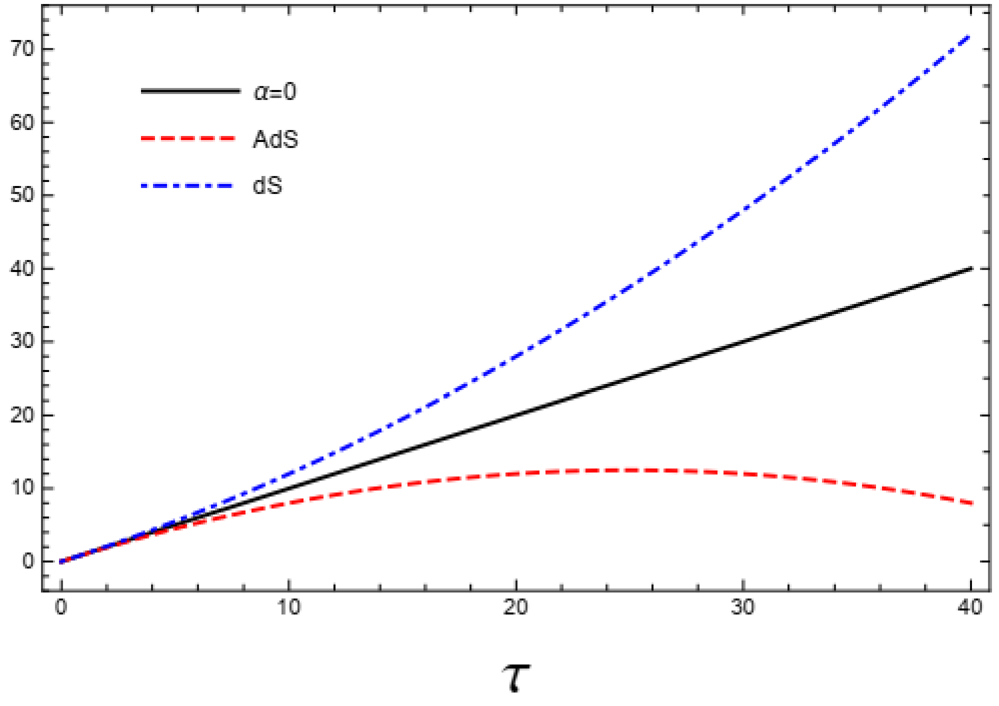

Figure 5: The classical internal energy versus τ.

In the EUP deformation scenario we observe that at higher temperature values the corrections increase the internal energy in the dS background, while they decrease the internal energy in the AdS background. We also notice that in the AdS space the internal energy tends to go to zero when the reduced temperature exceeds the critical temperature value.

Finally, we depict the specific heat function versus the reduced temperature in Fig. 6.

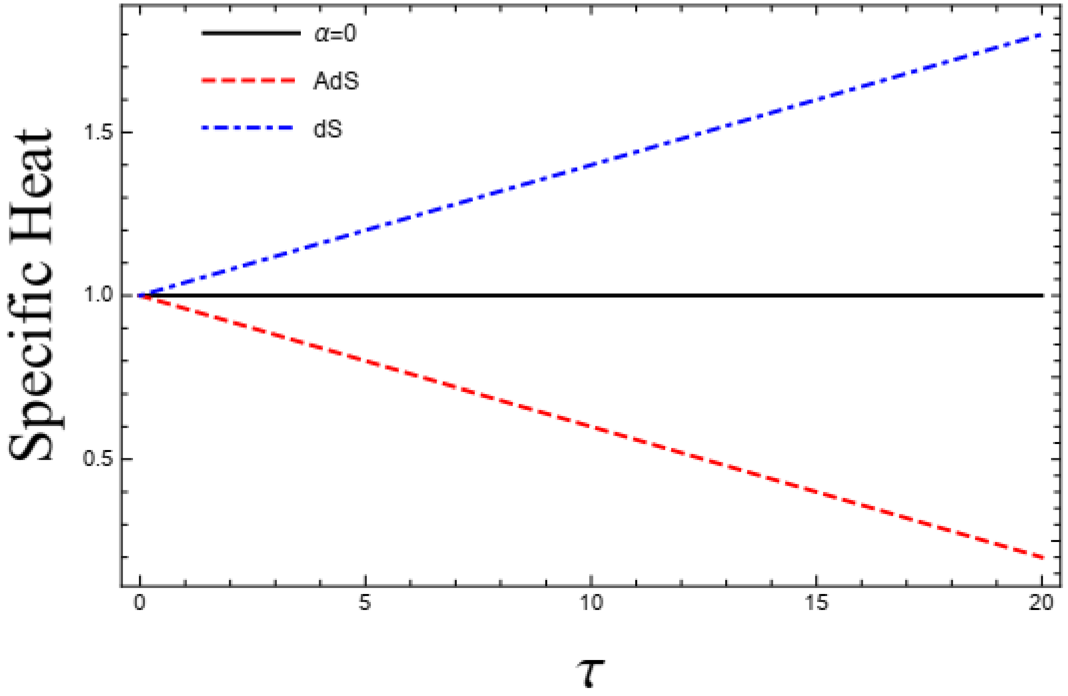

Figure 6: The classical specific heat versus τ.

the specific heat function increases linearly in the dS background, while it decreases linearly as the temperature increases in the AdS background.