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Proceeding Paper

Cosmological Solutions of Integrable $F(R)$ Gravity Models with an Additional Scalar Field †

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Abstract: We consider $F(R)$ cosmological models with scalar fields that can be transformed into two-field chiral cosmological models by the conformal metric transformation. Considering the R^2 model in the spatially flat Friedmann–Lemaître–Robertson–Walker metric, we have found that the Ricci scalar R can smoothly change its sign during the evolution if and only if the scalar field is a phantom one. In the Bianchi I metric, the Ricci scalar cannot smoothly change its sign if the corresponding solution is anisotropic at $R = 0$. The result in the Bianchi I metric does not depend on the type of the scalar field. In the Bianchi I metric, the general solutions have been obtained both for the R^2 gravity model with a scalar field without a potential and for the corresponding two-field chiral cosmological model.

Keywords: modified gravity; cosmology; exact solutions

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1. Modified Gravity Models and GR Models with Scalar Fields

On the one hand, $F(R)$ gravity generalizes the General Relativity (GR). On the other hand, gravitational $F(R)$ models can be considered as models with a nonminimally coupled scalar field without kinetic term. These models can be transformed into GR models with a standard minimally coupled scalar field by the conformal metric transformation. In other words, these models in the Einstein frame are GR models with one minimally coupled scalar field having a standard kinetic part. Note that such a transformation is possible only if the first derivative $F_{,R} \equiv \frac{dF}{dR} > 0$. A GR model with an ordinary scalar field has is well-defined at any value of R, whereas for $F(R)$ models, the possible domain of values of R is restricted by requirement the effective gravitational constant should be positive, hence, $F_{,R} > 0$. It is interesting to analyze whether $F_{,R}$ can smoothly change its sign during evolution or not. For models with a nonminimally coupled scalar field, this problem has been discussed in [1]. For $F(R)$ models it has been shown [2], that cosmological solutions have anisotropic instabilities associated with the crossing of the barrier $F_{,R}(R) = 0$. In other words, the solutions in the the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric can be smooth, whereas solutions in the Bianchi I metric should have singularities. These results have been obtained for $F(R)$ models without scalar fields.

The simplest modifications of GR that allow to describe inflation are the $R + R^2$ Starobinsky model [3] and the Higss-driven inflation [4]. These both models are in a good agreement to the Planck measurements of the Cosmic Microwave Background (CMB) radiation [5]. Inflationary R^2 models gravity with the Higgs-like boson are actively investigated in context of the possible production of primordial black holes [6–8]. Note that the R^2 term arises as a quantum correction in inflationary models with scalar fields [9–12].

The Starobinsky inflationary model [3,13] includes both R^2 term and the standard Hilbert-Einstein term. Adding to this model the cosmological constant, one can obtain a model with exact cosmological solutions [14]. The integrability of the Starobinsky model as well as the integrability of the pure R^2 model have been shown by the singularity analysis [15,16]. These models pass the so-called weak Painlevé test [16,17]. At the same time, exact cosmological solutions for the Starobinsky inflationary model are not known.

Most of the results of the explicit integration of cosmological models with scalar fields are related to one-field cosmological models and the spatially flat FLRW metric [18–22]. Anisotropic cosmological solutions in $F(R)$ gravity models have been actively investigated in Refs. [23–26].

In our papers [27,28], we analyze $F(R)$ gravity models with an additional scalar field. Such models as well as models with multiple scalar fields nonminimally coupled with gravity cannot be transformed to models with a few minimally coupled standard scalar fields in the generic case [29]. After the metric transformation, models with a nonstandard kinetic part of scalar field Lagrangian, the so-called chiral cosmological models, are obtained [30–36]. We are looking for general solutions of evolutionary equations in the pure R^2 model with a scalar field in the FLRW and Bianchi I metrics, using the corresponding chiral cosmological model. The absence of the scalar field potential in the model considered allows us to get the behaviour of the Hubble parameter in the analytic form, namely, in the form of hyperbolic tangent or cotangent. We also obtained scalar fields in the analytic form [27,28]. The knowledge of this general solution allows us to get the general solution of the initial $F(R)$ gravity model with an additional standard or phantom scalar field in the parametric time. After this, using the time transformation, we get solutions in the cosmic time in quadratures. This method allows to find only solutions with $F_{,R} > 0$, so it is important to analyze the possibility the existence of solutions with $F_{,R}$ changing its sign during the evolution.

2. Equations and Solutions in the Bianchi I and FLRW Metrics

Let us consider a pure R^2 model, describing by the following action:

$$S_R = \int d^4x \sqrt{-g}\left[F_0 R^2 - \frac{\varepsilon_\psi}{2} g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - V(\psi)\right], \tag{1}$$

where $F_0 > 0$ is a constant, R is the Ricci scalar, ψ is a scalar field or a phantom scalar field in dependence of the sign of $\varepsilon_\psi = \pm 1$.

Let us consider the case of the Bianchi I metric with the following interval [23,26,37,38]:

$$ds^2 = -dt^2 + a^2(t)\left[e^{2\beta_1(t)}\,dx_1^2 + e^{2\beta_2(t)}\,dx_2^2 + e^{2\beta_3(t)}\,dx_3^2\right]. \tag{2}$$

The functions $\beta_i(t)$ satisfy the relation

$$\beta_1(t) + \beta_2(t) + \beta_3(t) = 0. \tag{3}$$

It is useful to introduce a shear,

$$\sigma^2 \equiv \dot\beta_1^2 + \dot\beta_2^2 + \dot\beta_3^2 = 2\left(\dot\beta_1^2 + \dot\beta_1\dot\beta_2 + \dot\beta_2^2\right), \tag{4}$$

that measures a total amount of anisotropy, "dots" denote derivatives with respect to time t. The Ricci scalar is

$$R = \sigma^2 + 6\left(\dot H + 2H^2\right), \qquad H = \frac{\dot a}{a}. \tag{5}$$

The evolution equations in the Bianchi I metric have the following form [26,28]:

$$4H\dot\sigma^2 - \left(\sigma^2\right)^2 - 4\left(2\dot H + 3H^2\right)\sigma^2 + 24H\ddot H - 12\dot H^2 + 72H^2\dot H = \frac{3\varepsilon_\psi}{2F_0}\left(\dot\psi^2 + 2V(\psi)\right), \tag{6}$$

$$
\begin{aligned}
&-\ddot{\sigma}^2-2H\dot{\sigma}^2-\frac{1}{4}\left(\sigma^2\right)^2-\left(2\dot{H}+3H^2\right)\sigma^2\\
&+\left(6\dot{H}+12H^2+\sigma^2\right)\ddot{\beta}_i+\left(6\ddot{H}+42H\dot{H}+36H^3+3H\sigma^2+\dot{\sigma}^2\right)\dot{\beta}_i\\
&-3\left(2\dddot{H}+12H\ddot{H}+9\dot{H}^2+18H^2\dot{H}\right)=\frac{\varepsilon_\psi}{8F_0}\left(\dot{\psi}^2-2V(\psi)\right),\qquad i=1,2,3.
\end{aligned}
\tag{7}
$$

If $V(\psi)=0$, then we can eliminate $\dot{\psi}$ and obtain the following equations

$$
\frac{1}{6}\left(\ddot{\sigma}^2+5H\dot{\sigma}^2-2\left(2\dot{H}+3H^2\right)\sigma^2-\frac{\left(\sigma^2\right)^2}{2}\right)+\dddot{H}+9H\ddot{H}+3\dot{H}^2+18H^2\dot{H}=0, \tag{8}
$$

$$
\dot{\sigma}^2+\left(3H+\frac{\dot{R}}{R}\right)\sigma^2=0. \tag{9}
$$

Therefore, the evolution equations can be presented in the form of the following fifth-order system [28]:

$$
\dddot{H}=\frac{1}{2R}\left(r_1-2r_2\left[2\sigma^2+R\right]\right), \tag{10}
$$

$$
\ddot{\sigma}^2=\frac{3}{R}\left(4\sigma^2r_2-r_1\right), \tag{11}
$$

where

$$
r_1=\left(\dot{\sigma}^2\right)^2+\left(4H\sigma^2+6\ddot{H}+36H\dot{H}+24H^3\right)\dot{\sigma}^2+2\left(\dot{H}\sigma^2+14H\ddot{H}+14\dot{H}^2+36H^2\dot{H}\right)\sigma^2,
$$

$$
r_2=\frac{1}{12}\left(10H\dot{\sigma}^2-4\sigma^2\left(2\dot{H}+3H^2\right)-\left(\sigma^2\right)^2\right)+9H\ddot{H}+3\dot{H}^2+18H^2\dot{H}.
$$

Note that Equation (9) includes neither $\dddot{H}$, nor $\ddot{\sigma}^2$. The initial conditions for system (10)–(11) should satisfy the relation (9).

One can see, that the right hand sides of Equations (10) and (11) are singular at $R=0$ if $\sigma^2\neq 0$. Therefore, we see essential difference between anisotropic solutions, for which the sign of R is defined by the initial conditions and cannot be smoothly changed during evolution, and isotropic solutions that are solutions in that spatially flat FLRW metric. The existence of isotropic solutions that can smoothly change its sign during the evolution has been proved in [27], where the following explicit example has been found.

In the spatially flat FLRW metric, all $\beta_i(t)\equiv 0$ and, hence $\sigma^2(t)\equiv 0$. At $\sigma^2(t)\equiv 0$, Equations (9) and (11) are satisfied. The Hubble parameter $H(t)$ is a solution of Equation (10) that has the following form:

$$
\dddot{H}=r_2=9H\ddot{H}+3\dot{H}^2+18H^2\dot{H}. \tag{12}
$$

This equation is equivalent to the following equation:

$$
\left(\ddot{H}+3H\dot{H}\right)\frac{d}{dt}\left[\dot{H}^2-2H\ddot{H}-6H^2\dot{H}\right]=\left(\dot{H}^2-2H\ddot{H}-6H^2\dot{H}\right)\frac{d}{dt}\left[\ddot{H}+3H\dot{H}\right]. \tag{13}
$$

Equation (13) has been analyzed in detail in our paper [27], where two families of solutions have been found. The Hubble parameter $H(t)$ is either a solution of equation

$$
\ddot{H}+3H\dot{H}=0, \tag{14}
$$

or a solution of equation

$$
\dot{H}^2-2H\ddot{H}-6H^2\dot{H}=0. \tag{15}
$$

Equation (14) can be easily integrated and we get

$$2\dot{H}+3H^2=2C,$$

where C is an integration constant.

If $C<0$, then the following exact solution exists:

$$H(t)=-\frac{\sqrt{-6C}}{3}\tan\left[\frac{\sqrt{-6C}}{2}(t-t')\right],\quad \psi(t)=\frac{6C\sqrt{2F_0}}{\cos^2\left(\sqrt{\frac{-3C}{2}}(t-t')\right)}. \tag{16}$$

Note that such solutions exist only if $\varepsilon_\psi=-1$, because

$$\dot{\psi}^2=-72\varepsilon_\psi F_0\dot{H}^2. \tag{17}$$

So, the scalar field ψ is a phantom one. Substituting $\sigma^2=0$ and $H(t)$, given by (16) into (5), we get that the scalar curvature R changes its sign at $t=t'\pm\frac{\pi}{3}\sqrt{-2/(3C)}$. Therefore the solution obtained has no analogue in the Einstein frame. It is one of main results of our paper [27].

3. Conclusions

Modified gravitational models are being actively investigated both to describe the accelerating expansion of the observed Universe and to link general relativity with quantum theory due to adding of string theory inspired corrections [39,40]. It is an important issue to obtain cosmological solutions for such types of models. The standard way to explore $F(R)$ gravity models and models with non-minimally coupled scalar field is to use the conformal metric transformation and to analyze the corresponding GR model in the Einstein frame [22,41]. However, one can lose some solutions by this way. For example, if $F_{,R}$ and, therefore, the effective gravitational constant change their signs during the evolution, then the corresponding smooth solutions for $F(R)$ gravity models has no analogue in the Einstein frame. It is important to check independently the existence of such solutions analyzing evolution equations in the original frame.

We have found smooth particular solutions, in particular, solution (16), which have no analogue in the Einstein frame, because the Ricci scalar R changes its sign during the evolution. Note that these solutions do not admit any amount of anisotropy. In the Bianchi I metric, the evolution Equations (10) and (11) have singularities at $R=0$ for anisotropic solutions. So, for anisotropic solutions, the Ricci scalar R does not change its sign during evolution.

We have found general solutions in the spatially flat FLRW and Bianchi I metrics both for the R^2 gravity model with an additional scalar field and for the corresponding two-field chiral cosmological model [27,28]. The cosmic time in the Einstein frame corresponds to a parametric time in the Jordan frame, so, the consideration of integrable chiral cosmological models is useful to get the general solutions for the corresponding modified gravity models in parametric time. Solutions in the cosmic time for such models can be found in quadratures.

The proposed method allows to find general solutions for different modified gravity cosmological models that can be transformed into chiral cosmological models.

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