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Bianchi Type-I Universe in Modified Theory of Gravity†

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Abstract: In this paper, we have studied an anisotropic Bianchi-I cosmological model in $f(R, T)$ gravity. To obtain exact solutions of the field equations, we have used the condition σ/θ to be a function of the scale factor (**IJTP, 54, 2740-2757, 2015**). Our model posseses an initial singularity. It initially exhibits decelerating expansion and transits to accelerating expansion at late times. We have also discussed the physical and geometrical properties of the model.

Keywords: Bianchi Type-I; variable anisotropy parameter; equation of state parameter.

1. Introduction

Undoubtedly, Einstein's general theory of relativity (GR) is the most widely accepted theory to study the evolution of universe. Early on, before expanding, the universe was in a very small, hot, and dense state. In the light of recent observations of type Ia supernova, it was found that the expansion of universe is in fact accelerating [1,2]. This theory of the accelerating expansion of the universe has been supported by recent observational data [3,4]. For vindicating this acceleration, a new energy that exhibits repulsive force was postulated, it is named Dark Energy (DE) [5–7]. The universe is made up of three forms of matter/energy, viz., 68.5% DE, 26.6% dark matter and only 4.9% baryonic matter. This is in accordance with observations made by the Planck mission team [6].

DE can be quintessence, phantom, k-essence, tachyons, quintom, Chaplygin gas, chameleon and the cosmological constant (CC). The second alternative is a modification of GR. The former is problematic, as the usage of the CC comes with its own drawbacks in the form of theoretical problems such as cosmic coincidence and fine-tuning.

Taking another approach, the cause for the acceleration of the universe can be sought in modified gravity theories. Harko, et al. [8,9] introduced $f(R, T)$ gravity, in which $f(R, T)$ is a function of T and R, where T is the trace of the energy momentum tensor and R is the Ricci scalar. This theory permits an explanation of accelerated expansion without DE, and for the circumvention of the initial singularity. There is a violation of the usual energy conservation [10], but one achieve this with the aid of another condition. It permits for a dynamical cosmological parameter [11]. This aids in the likelihood of solving the cosmological constant problem.

In this article, we have studied a cosmological model in $f(R, T)$ gravity for a Bianchi-I universe with perfect fluid. We have discussed a cosmological model which isotropizes at late times.

2. Review of $f(R, T)$ and field equations

The gravitational action for $f(R, T)$ takes the following form:

$$S = \int \left(\frac{1}{16\pi} f(R, T) + S_m \right) \sqrt{-g} dx^4, \tag{1}$$

We assume that [8]

$$f(R, T) = R + 2h(T), \tag{2}$$

and put $h(T) = \alpha T$, where α is called the coupling constant of $f(R, T)$ gravity. Equation (2) can be rewritten as

$$f_R(R, T)R_{ij} - \frac{1}{2} f(R, T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R(R, T) = 8\pi T_{ij} - f_T(R, T)T_{ij} - f_T(R, T)\Theta_{ij}. \tag{3}$$

Here $\Box$ is D'Alembertian operator which is expressed as ($\Box = \nabla^i\nabla_j$), and Θ_{ij} is expressed as

$$\Theta_{ij} = g^{lm} \frac{\delta T_{lm}}{\delta g^{ij}}. \tag{4}$$

Let us consider that the matter content in the universe is a perfect fluid and that

$$\Theta_{ij} = -2T_{ij} - pg_{ij}, \tag{5}$$

where T_{ij} is energy momentum tensor

$$T_{ij} = (\rho + p)u_iu_j - pg_{ij}, \tag{6}$$

where ρ and p are the energy density and cosmic pressure. u^i is four velocity vector such that $u^iu_i = 1$. The field equations then take the form

$$R_{ij} - \frac{1}{2} Rg_{ij} = 8\pi T_{ij} + 2f_T T_{ij} + [f(T) + 2pf_T]g_{ij}. \tag{7}$$

The B-I metric is:

$$ds^2 = -dt^2 + X^2 dx^2 + Y^2 dy^2 + Z^2 dz^2, \tag{8}$$

where X, Y and Z are function of time t only. Then from equations (7) and (2), we get:

$$\dot{H}_y + H_y^2 + \dot{H}_z + H_z^2 + H_yH_z = -(8\pi + 3\alpha)p + \alpha\rho, \tag{9}$$

$$\dot{H}_x + H_x^2 + \dot{H}_z + H_z^2 + H_xH_z = -(8\pi + 3\alpha)p + \alpha\rho, \tag{10}$$

$$\dot{H}_x + H_x^2 + \dot{H}_y + H_y^2 + H_xH_y = -(8\pi + 3\alpha)p + \alpha\rho, \tag{11}$$

$$H_xH_y + H_yH_z + H_zH_x = (8\pi + 3\alpha) - \alpha p, \tag{12}$$

where $H_x = \frac{\dot{X}}{X}$, $H_y = \frac{\dot{Y}}{Y}$ and $H_z = \frac{\dot{Z}}{Z}$. The average scale factor (a) is

$$a^3 = XYZ. \tag{13}$$

Using equations (9) and (10), we get

$$(\dot{H}_x + H_x^2) - (\dot{H}_y + H_y^2) - H_z(H_x - H_y) = 0. \tag{14}$$

After integration, we get

$$H_x - H_y = \frac{k_1}{a^3}. \tag{15}$$

Again, by using equations (10) and (11), we obtain

$$H_y - H_z = \frac{k_2}{a^3}, \tag{16}$$

where k_1 and k_2 are constants of integration.

From equations (13), (15) and (16), we obtain

$$H_x = \frac{\dot{a}}{a} + \frac{(2k_1 + k_2)}{3a^3}, H_y = \frac{\dot{a}}{a} + \frac{(k_2 - k_1)}{3a^3}, H_z = \frac{\dot{a}}{a} - \frac{(k_1 + 2k_2)}{3a^3}. \tag{17}$$

The Hubble parameter (H), expansion scalar (θ), deceleration parameter (q) and shear scalar (σ) are defined as:

$$\frac{1}{3}\theta = H = \frac{1}{3}(H_x + H_y + H_z), \quad q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1, \quad \sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}, \tag{18}$$

Then we have

$$\sigma = \frac{k}{a^3}, \tag{19}$$

where $3k^2 = k_1^2 + k_2^2 + k_1 k_2$. Now, the pressure p and energy density ρ, can be written as:

$$p = \frac{-1}{[(8\pi + 3\alpha)^2 - \alpha^2]}\left[2(8\pi + 3\alpha)\frac{\ddot{a}}{a} + 8\pi\left(\frac{\dot{a}}{a}\right)^2 + \frac{(8\pi + 4\alpha)k^2}{a^6}\right], \tag{20}$$

$$\rho = \frac{1}{[(8\pi + 3\alpha)^2 - \alpha^2]}\left[-2\alpha\frac{\ddot{a}}{a} + (24\pi + 8\alpha)\left(\frac{\dot{a}}{a}\right)^2 - \frac{(8\pi + 4\alpha)k^2}{a^6}\right]. \tag{21}$$

3. Solutions and physical interpretation

In order to get solutions, let us require σ/θ to be a function of $a(t)$ such that σ/θ is very large at early times, and that tends to zero for large t. Hence, we choose

$$\frac{\sigma}{\theta} = \frac{k}{3a^2\dot{a}} = f(a). \tag{22}$$

After integration, we get

$$k(t + t_1) = 3 \int a^2 f(a) da, \tag{23}$$

where t_1 is a constant of integration. Consider the function $f(a)$ as [12]:

$$f(a) = \frac{1}{(n + a^3)^m}, \tag{24}$$

where m and n are positive integers. Our motivation for choosing this form of $f(a)$ is as follows. The average anisotropy is $\bar{A} = 6\sigma^2/\theta^2$. Now, we want a a cosmological model in which the anisotropy decreases with time and asymptotically approaches zero. One possible way to achieve this is to take the form (24) which satisfies this requirement. For this consideration, we obtain at early times $\sigma/\theta \neq 0$ and at late times $\sigma/\theta \to 0$. By using equations (23) and (24), we get

$$k(t + t_1) = 3 \int a^2 (n + a^3)^{-m} da. \tag{25}$$

We discuss the solutions for $m = 1$ and $m \neq 1$ separately, taking $t_1 = 0$ without loss of generality.

3.1. $m = 1$

On integrating the above equation, we get

$$a = [n(e^{kt} - 1)]^{\frac{1}{3}}. \tag{26}$$

The expansion scalar (θ), shear scalar (σ) and deceleration parameter (q) are given by:

$$H = \frac{k}{3(1 - e^{-kt})}, \quad \theta = \frac{k}{1 - e^{-kt}}, \quad \sigma = \frac{k}{n(e^{kt} - 1)}, \quad q = -1 + 3e^{-kt}. \tag{27}$$

These can be plotted to observe their behaviour, and will be done elsewhere. The Hubble parameter $H(t)$ is finite for all finite values of $t > 0$, and $\omega > -1$ for all finite values of $t > 0$. The scale factor (a) is zero and the expansion scalar (θ) diverges $t = 0$. The model possesses an initial singularity . The deceleration parameter (q) indicates deceleration at early times, and acceleration at late times. In this model, the deceleration parameter lies between 2 to -1.

The anisotropy parameter is:

$$\frac{\sigma}{\theta} = \frac{1 - e^{-kt}}{n(e^{kt} - 1)}. \tag{28}$$

We see that σ/θ is non-zero and finite at $t = 0$, and $\sigma/\theta \to 0$ as $t \to \infty$. Thus the model approaches late time isotropy. The pressure and energy density are given by:

$$p = \frac{-1}{[(8\pi + 3\alpha)^2 - \alpha^2]} \left[\frac{2(8\pi + 3\alpha)k^2(1 - 3e^{-kt})}{9(1 - e^{-kt})^2} + \frac{8\pi k^2}{9(1 - e^{-kt})^2} + \frac{(8\pi + 4\alpha)k^2}{[n(e^{kt} - 1)]^2} \right], \tag{29}$$

$$\rho = \frac{1}{[(8\pi + 3\alpha)^2 - \alpha^2]} \left[\frac{-2\alpha k^2(1 - 3e^{-kt})}{9(1 - e^{-kt})^2} + \frac{(24\pi + 8\alpha)k^2}{9(1 - e^{-kt})^2} - \frac{(8\pi + 4\alpha)k^2}{[n(e^{kt} - 1)]^2} \right]. \tag{30}$$

Here we observe that energy density is a positive and decreasing function of t. The equation of state parameter $(\omega = p/\rho)$ is given by:

$$\omega = -\frac{\left[\frac{2(8\pi + 3\alpha)k^2(1 - 3e^{-kt})}{9(1 - e^{-kt})^2} + \frac{8\pi k^2}{9(1 - e^{-kt})^2} + \frac{(8\pi + 4\alpha)k^2}{[n(e^{kt} - 1)]^2} \right]}{\left[\frac{-2\alpha k^2(1 - 3e^{-kt})}{9(1 - e^{-kt})^2} + \frac{(24\pi + 8\alpha)k^2}{9(1 - e^{-kt})^2} - \frac{(8\pi + 4\alpha)k^2}{[n(e^{kt} - 1)]^2} \right]}. \tag{31}$$

The energy density $\rho \to \infty$ as $t \to 0$ and it approaches zero at late times. The pressure p is negative at late times, and this can be associated with late time acceleration. The equation of state parameter ω is positive at the beginning, and it approaches -1 as $t \to \infty$. The present value of ω of our model is nearly -0.873 which is less than -1.

3.2. $m \neq 1$

In this case equation (26) becomes:

$$a = \left[\left\{ n^{1-m} + (1 - m)kt \right\}^{\frac{1}{1-m}} - n \right]^{\frac{1}{3}}. \tag{32}$$

The scale factor a is increasing for $0 < m < 1$. The Hubble parameter (H), shear scalar (σ) and deceleration parameter (q) are:

$$H = \frac{k\{n^{1-m} + (1-m)kt\}^{\frac{m}{1-m}}}{3\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]},$$
(33)

$$\sigma = \frac{k}{\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]},$$
(34)

$$q = 2 - 3m\left[1 - n\{n^{1-m} + (1-m)kt\}^{\frac{-1}{1-m}}\right].$$
(35)

We note that the Hubble parameter $H(t)$ is finite for all finite values of $t > 0$, and $\omega > -1$ for all finite values of $t > 0$.

The pressure and energy density are:

$$p = \frac{-1}{[(8\pi + 3\alpha)^2 - \alpha^2]}$$

$$\left[\frac{2(8\pi + 3\alpha)k^2\left[(3m+1)\{n^{1-m} + (1-m)kt\}^{\frac{2m}{1-m}} - 3(mn+1)\{n^{1-m} + (1-m)kt\}^{\frac{2m-1}{1-m}}\right]}{9\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]^2}\right.$$

$$\left. + \frac{8\pi k^2\{n^{1-m} + (1-m)kt\}^{\frac{2m}{1-m}}}{9\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]^2} + \frac{(8\pi + 4\alpha)k^2}{\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]^2}\right],$$
(36)

$$\rho = \frac{1}{[(8\pi + 3\alpha)^2 - \alpha^2]}$$

$$\left[\frac{-2\alpha k^2\left[(3m+1)\{n^{1-m} + (1-m)kt\}^{\frac{2m}{1-m}} - 3(mn+1)\{n^{1-m} + (1-m)kt\}^{\frac{2m-1}{1-m}}\right]}{9\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]^2}\right.$$

$$\left. + \frac{(24\pi + 8\alpha)k^2\{n^{1-m} + (1-m)kt\}^{\frac{2m}{1-m}}}{9\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]^2} - \frac{(8\pi + 4\alpha)k^2}{\left[\{n^{1-m} + (1-m)kt\}^{\frac{1}{1-m}} - n\right]^2}\right],$$
(37)

The model has a singularity at $t = 0$. The deceleration parameter is initially positive and negative at late times for $\frac{2}{3} < m < 1$. Again, the universe is initially decelerating and accelerating at late times. The anisotropy parameter is:

$$\frac{\sigma}{\theta} = \frac{1}{\left[\{n^{1-m} + (1-m)kt\}^{\frac{m}{1-m}}\right]}$$
(38)

Again, We see that σ/θ is non-zero and finite at $t = 0$, and $\sigma/\theta \to 0$ as $t \to \infty$. Thus the model approaches late time isotropy.

The energy density $\rho \to \infty$ as $t \to 0$ and it approaches zero at late times. The pressure p is negative. Negative pressure is associated with the late time accelerating expansion of the universe.

4. Conclusions

In this paper, we discussed at B-I cosmological models with perfect fluid in a modified theory of gravity. In order to find solutions of the modified field equations, we assumed that σ/θ ia a suitable function of the scale factor a. While exploring this case, we found that

the model begins with a big bang. The model is initially decelerating, and them switches to acceleration later on.

The key findings of the model are listed below:

• It is observed that the projected variation law for the expansion anisotropy σ/θ imparts an alternate approach to obtain exact solutions of the field equations. It may eliminate earlier inconsistencies of models obtained by assuming σ/θ to be constant.

• It is noted that the Hubble parameter is a decreasing function of time t tends to zero .

• The deceleration parameter changes from positive early on to negative later on, i.e., a change from early deceleration to late time acceleration. The deceleration parameter lies between -1 and 2 in our model.

• The energy density of the cosmic fluid remains positive throughout the cosmic expansion. Nevertheless, the cosmic pressure is positive early on, and negative at late times. The present accelerated expansion of the universe could be due to these negative values.

• The equation of state parameter also supports observational data.

The outcomes from our model concur with modern observational data. We hope to report on the observational constraints at a later stage. The proposed model extends a beneficial benchmark in the analysis of B-I cosmological models with perfect fluid in a modified theory of gravity.

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