Calculation of the Vacuum Energy Density using Zeta Function Regularization

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Outline

- Vacuum energy (density)
- Riemann Zeta function and its application to physics
- A new Zeta function regularization technique
- Zeta function regularization applied to vacuum energy problem
- Results
- Discussion and Conclusions

Vacuum Energy

The vacuum (zero-point) energy of a free (non-interacting) quantum field in flat spacetime, can be modeled as the sum of zero-point energies of a set of infinite number of quantum harmonic oscillators, one for each normal mode and given simply as $+\sum_k \frac{1}{2}\hbar\omega_k$ for bosonic fields and $-\sum_k \frac{1}{2}\hbar\omega_k$ for fermionic fields [3], [4], and where $\frac{1}{2}\hbar\omega_k$ represents the eigenvalues of the free Hamiltonian, $\omega_k = \sqrt{m^2 + k^2}$ in Natural units, k is the wave number (with units of 1/length), m is the particle mass associated with a specific field (assumed to be constant and not running), and $\hbar$ is the reduced Planck constant. The negative sign, in front of the infinite sum, is due to the negative energy solutions allowed in the fermionic fields which must follow the Pauli Exclusion principle by obeying the canonical anti-commutation relations.

[3] W. Pauli, "Pauli Lectures on Physics: Vol 6, Selected Topics in Field Quantization", MIT Press, 1971.

[4] R. D. Klauber, "The Student Friendly Quantum Field Theory", 2013. https://www.quantumfieldtheory.info/

Vacuum Energy (density)

$$\overline{\rho}_{vac} = \frac{E}{V} = \pm\frac{1}{V}\sum_k \frac{1}{2}\hbar\omega_k = \ldots = \pm\frac{1}{4\pi^2}\int_0^\infty \sqrt{m^2+k^2}k^2 dk$$

$$\overline{\rho}_{vac} = g\frac{1}{4\pi^2}\int_0^\infty \sqrt{(mc^2)^2+(\hbar kc)^2}k^2 dk$$

where g is the degeneracy factor which includes, a sign factor $(-1)^{2j}$, a spin factor $2j+1$ for massive fields and 2 for massless fields, a factor of 3 for fields with color charge, and a factor of 2 for fields that have an antiparticle distinct from their particle. Note also that c is the speed of light in vacuum and j is the spin. It is also important to note that Pauli [4] introduced the following 3 polynomial-in-mass and 1 logarithmic-in-mass conditions in order to obtain a vanishing vacuum energy density:

$$\sum_i g_i m_i^4 = 0 \qquad \sum_i g_i m_i^2 = 0 \qquad \sum_i g_i = 0 \qquad \sum_i g_i m_i^4 \ln\left(\frac{m_i^2}{\mu^2}\right) = 0$$

Vacuum Energy (density) catastrophe!

$$\bar{\rho}_v = \frac{1}{2\pi^2}\int_0^\Lambda \sqrt{m^2+k^2}\,k^2 dk \approx \frac{1}{2\pi^2}\int_0^\Lambda k^3 dk = \frac{\Lambda^4}{8\pi^2} \tag{10-22}$$

In natural units, the Planck mass ≈ 1.22×10^{19} GeV. Thus, the theoretical value for energy density of the vacuum is

$$\bar{\rho}_v \approx \frac{\left(1.22\times10^{19}\right)^4}{8\pi^2} = 2.80\times10^{74}\ \text{GeV}^4 . \tag{10-23}$$

Note length in natural units is GeV^{-1}, so an energy per unit volume would be measured in GeV^4. The experimental value for the upper limit on energy density of the vacuum is

$$\bar{\rho}_v \leq 10^{-47}\ \text{GeV}^4 , \tag{10-24}$$

a discrepancy between theory and experiment by a factor of more than 10^{120}. Yikes.

[6] R. D. Klauber, "The Student Friendly Quantum Field Theory", 2013. https://www.quantumfieldtheory.info/

Vacuum Energy density

$$\overline{\rho}_{vac} = g\frac{1}{4\pi^2}\int_0^\infty \sqrt{(mc^2)^2 + (\hbar kc)^2}k^2dk$$

$$\overline{\rho}_{vac} = g\frac{\hbar c}{4\pi^2}\int_0^\infty \sqrt{\left(\frac{mc}{\hbar}\right)^2 + k^2}k^2dk = g\frac{\hbar c}{4\pi^2}\int_0^\infty \sqrt{a^2 + k^2}k^2dk \qquad a = \frac{mc}{\hbar}$$

$$= g\frac{\hbar c}{4\pi^2}\int_0^\infty a\sqrt{1 + \left(\frac{k}{a}\right)^2}k^2dk \qquad x = k/a \text{ and } dx = dk/a$$

$$\overline{\rho}_{vac} = g\frac{\hbar c}{4\pi^2}\int_0^\infty a\sqrt{1 + x^2}(ax)^2adx = g\frac{\hbar c}{4\pi^2}a^4\int_0^\infty \sqrt{1 + x^2}x^2dx$$

$$= g\frac{m^4c^5}{4\pi^2\hbar^3}\int_0^\infty \sqrt{1 + x^2}x^2dx$$

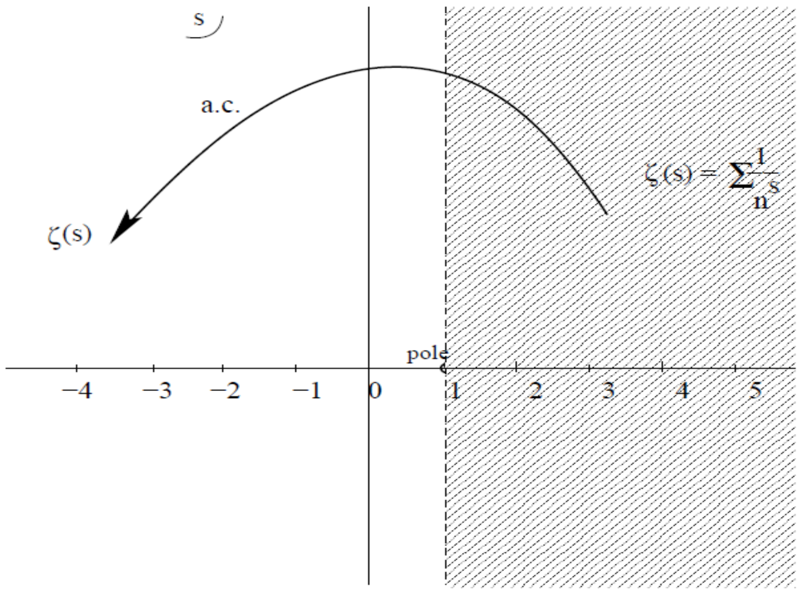
We will return to this integral!

Riemann Zeta function and its application to physics

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$S_1 = 1+1+1+1+\cdots \qquad S_1 = \zeta(0) = -\frac{1}{2}$$

$$S_2 = 1+2+3+4+\cdots \qquad S_2 = \zeta(-1) = -\frac{1}{12}$$



- Hawking S.W. (1977), Zeta function regularization of path integrals in curved spacetime, *Communications in Mathematical Physics, 55* (2) 133-148.

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- Cassimir Effect: zeta(-3) = 1/120

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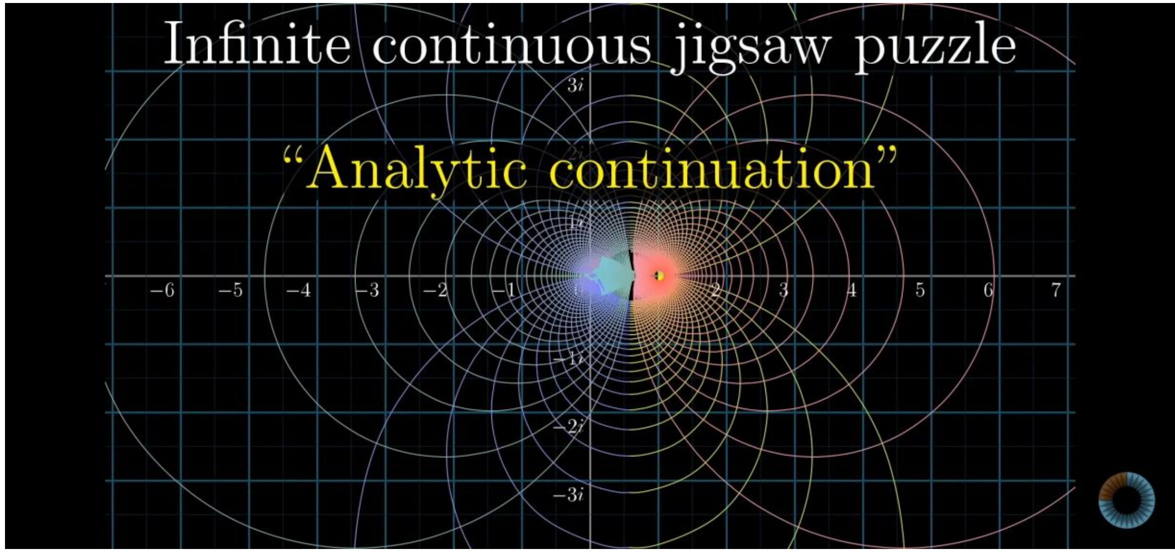
- Remmen G. N. (2021), Amplitudes and the Riemann Zeta Function, arXiv:2108.07820v2



3Blue1Brown – YouTube channel

A new Riemann Zeta function regularization technique

Let's define the μ function as the improper integral of a power function with exponent p

$$\mu(p) = \int_0^\infty x^p \, dx \tag{10}$$

with $p \in \mathbb{N}^0$, where $\mathbb{N}^0 = \{0, 1, 2, \cdots\}$.

The mu function can be interpreted as the natural extension of the Riemann zeta function where the discrete sum is replaced by a continuous integral. The above improper integral can be equivalently written as the following infinite series by splitting the limits of integration into successive integer numbers. That is

$$\mu(p) = \sum_{n=1}^{\infty} \Delta(p, n) \tag{11a}$$

where

$$\Delta(p, n) = \int_{n-1}^{n} x^p \, dx = \frac{1}{p+1}\left(n^{p+1} - (n-1)^{p+1}\right) \tag{11b}$$

is the definite integral of the power function over limits $n-1$ and n. In what follows, we will show that the divergent series in the RHS of (11a) can converge to a finite result!

A new Zeta function regularization technique

$$\mu(p) = \frac{1}{p+1}\sum_{n=1}^{\infty}\sum_{k=0}^{p}\binom{p+1}{k}(-1)^{p-k}n^k = \frac{1}{p+1}\left(\sum_{k=0}^{p}\binom{p+1}{k}(-1)^{p-k}\sum_{n=1}^{\infty}n^k\right) \qquad (14)$$

The last summation in RHS of (14) can be written in terms of the Reimann zeta function $\zeta(-k) = \sum_{n=1}^{\infty} n^k$ and therefore we arrive at

$$\mu(p) = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k}(-1)^{p-k}\zeta(-k) \qquad \forall p \in \mathbb{N}^0 \qquad (15)$$

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$$

$$\boxed{\mu(p) = \int_0^{\infty} x^p\, dx = \frac{(-1)^{p+1}}{(p+1)(p+2)} \qquad \forall p \in \mathbb{N}^0} \qquad (21)$$

$$\mu(0) = -\frac{1}{2} = \frac{(-1)^{0+1}}{(0+1)(0+2)}$$

$$\mu(1) = \frac{1}{6} = \frac{(-1)^{1+1}}{(1+1)(1+2)}$$

$$\mu(2) = -\frac{1}{12} = \frac{(-1)^{2+1}}{(2+1)(2+2)}$$

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Zeta function regularization applied to vacuum energy problem

$$\int_0^\infty \sqrt{1+x^2}x^2dx \qquad f(x) = x^2\sqrt{1+x^2}$$

$$f(x) = x^2 + \frac{x^4}{2} - \frac{x^6}{8} + \frac{x^8}{16} - \frac{5x^{10}}{128} + \frac{7x^{12}}{256} - \frac{21x^{14}}{1024} + O(x^{16}, x^{18}, x^{20}, \ldots)$$

$$\int_0^\infty x^2dx = -\tfrac{1}{12}, \int_0^\infty x^4dx = -\tfrac{1}{30}, \int_0^\infty x^6dx = -\tfrac{1}{56} \text{ and } \int_0^\infty x^8dx = -\tfrac{1}{90}$$

$$\int_0^\infty \sqrt{1+x^2}x^2dx \approx -\frac{1}{10}$$

negative value!

$$\overline{\rho}_{vac} = g\frac{m^4c^5}{4\pi^2\hbar^3}\int_0^\infty \sqrt{1+x^2}x^2dx = -g\frac{m^4c^5}{40\pi^2\hbar^3} \equiv -g\frac{m^4}{40\pi^2} \quad (Natural\ Units)$$

$$\overline{\rho}_{total} = \sum_i (\overline{\rho}_{vac})_i = -\frac{1}{40\pi^2}\sum_i g_i m_i^4$$

Results

Theoretical

Fields	Degeneracy factor g sign $\times$ spin $\times$ color $\times$ antiparticle	Mass (eV/c^2)	$\overline{\rho}_{vac}$ ($\mathrm{J/m^3}$)
Quarks (x 6): fermionic	$-1\times(2\times1/2+1)\times3\times2=-12$	178.31×10^9	$+6.29\times10^{44}$
Leptons (x 6): fermionic	$-1\times(2\times1/2+1)\times1\times2=-4$	1.90×10^9	$+2.70\times10^{36}$
W: bosonic	$1\times(2\times1+1)\times1\times2=6$	80.38×10^9	-1.30×10^{43}
Z : bosonic	$1\times(2\times1+1)\times1\times1=3$	91.19×10^9	-1.08×10^{43}
Higgs : bosonic	$1\times(2\times0+1)\times1\times1=1$	125.10×10^9	-1.27×10^{43}
Total			$+5.93\times10^{44}$

$2.85\times10^7\ \mathrm{GeV}^4$

Observational

[9] Planck Collaboration, "Planck 2018 results. VI. Cosmological parameters", 2020. `https://arxiv.org/abs/1807.06209` (document)

$$\rho_{vac}=\frac{\Lambda}{8\pi G}$$

$5.26\times10^{-10}\ \mathrm{J/m^3}$ ($\Lambda=(4.24\pm0.11)\times10^{-66}\ \mathrm{eV}^2$)

$\Lambda=4.84\times10^{-12}\ \mathrm{eV}^2$

Results

1- Vacuum energy density (VED) is finite without the need if a UV cut-off.

2- VED has a simple closed-form, quartic in particle rest mass. $\overline{\rho}_{vac} = -\frac{1}{40\pi^2}\sum_i g_i m_i^4$

3- Only 1 of Pauli's 4 conditions is necessary and sufficient to zero out the VED. $\sum_i g_i m_i^4 = 0$

4- Vacuum energy contributions from the fermionic fields are *positive* and from the bosonic field are *negative*!

Discussion and Conclusions

Why is there still a discrepancy between theoretical and observational results ?

1- We may be missing the vacuum energy contribution of at least one unknown massive bosonic particle. For example if we assume a scalar boson (say a heavy cousin of Higgs with g=1), then its mass would be **327 GeV/c^2** If we assume a vector boson (say a Z-prime boson with g=3), then its mass would be about **247 GeV/c^2** . Also it is possible there is a missing fermionic (dark matter) particle plus an even heavier missing boson.

2- The quantum harmonic oscillator model may be too simple to describe the vacuum energy. For example, the effect of interacting quantum fields may need to be taken into account (this work has started).

3- The gravitational field may have an important contribution. This will require an understanding of the quantum gravity which we currently lack.

4. The effect of running of the particle masses needs to be investigated (this work has started).

Backup Slides

10.8 Appendix A: Theoretical Value for Vacuum Energy Density

Given (10-14), the vacuum energy density of the ZPE is

$$\bar{\rho}_v = \frac{\bar{E}_v}{V} = \frac{1}{V}\langle 0|H_0^0|0\rangle = \frac{1}{V}\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(\tfrac{1}{2}+\tfrac{1}{2}\right)|0\rangle = \frac{1}{V}\sum_{\mathbf{k}} \omega_{\mathbf{k}} = \frac{1}{l_1 l_2 l_3}\sum_{\mathbf{k}} \omega_{\mathbf{k}} = \sum_n \frac{\omega_{\mathbf{k}_n}}{l_1 l_2 l_3} \qquad (10\text{-}16)$$

Where l_i is the length of the ith side of volume V inside of which the particle/waves are found, and we have used slightly different labeling in the last expression to suit our immediate needs. Boundary conditions on V give the wavelength of the nth wave (n = 1, 2, …) in the x^1 direction inside as $\lambda_{n\,1} = l_1/n$, with similar relations for the other two directions. The wave number $k_{n\,1}$ of the nth wave is thus

$$k_{n\,1} = \frac{2\pi}{\lambda_{n\,1}} = \frac{2\pi n}{l_1}. \qquad (10\text{-}17)$$

Defining $\Delta k_1 = k_{n\,1}/n$, we have, from (10-17),

$$\frac{1}{l_1} = \frac{k_{n\,1}}{2\pi n} = \frac{\Delta k_1}{2\pi}. \qquad (10\text{-}18)$$

Similar results hold for l_2 and l_3. Thus, (10-16) (with the relativistic expression for energy used in the last step below) is

$$\bar{\rho}_v = \sum_n \frac{\omega_{\mathbf{k}_n}}{l_1 l_2 l_3} = \sum_n \omega_{\mathbf{k}_n} \frac{\Delta k_1}{2\pi}\frac{\Delta k_2}{2\pi}\frac{\Delta k_3}{2\pi} = \sum_n \sqrt{m^2 + \mathbf{k}_n^{\,2}}\,\frac{\Delta k_1}{2\pi}\frac{\Delta k_2}{2\pi}\frac{\Delta k_3}{2\pi}. \qquad (10\text{-}19)$$

For l_1 very large, from (10-18), $\Delta k_1 \to dk_1$, with similar expressions for large l_2 and l_3. In the limit of large l_i (large volume V), (10-19) becomes

$$\bar{\rho}_v = \int_{-\infty}^{\infty} \sqrt{m^2 + \mathbf{k}^2}\,\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi} = \int_{-\infty}^{\infty} \frac{1}{(2\pi)^3}\sqrt{m^2 + \mathbf{k}^2}\,d^3k \qquad (10\text{-}20)$$

Using (9-9) on pg. 260, we find this becomes

$$\bar{\rho}_v = \int_0^{\infty} \frac{1}{(2\pi)^3}\sqrt{m^2 + k^2}\,4\pi k^2 dk = \frac{1}{2\pi^2}\int_0^{\infty} \sqrt{m^2 + k^2}\,k^2 dk\,. \qquad (10\text{-}21)$$

This is obviously infinite, unless we take an upper limit cutoff, typically considered the Planck scale mass (energy). This is because a particle with energy of the Planck mass is assumed to have an associated Compton wavelength (size of the particle) so small that its associated mass-energy forms a microscopic black hole. Spacetime is not defined inside a black hole, so smaller size (larger energy) particles may not be able to exist in our universe. Given such logic, with Λ = Planck mass $\gg m$, we find (10-21) is

$$\bar{\rho}_v = \frac{1}{2\pi^2}\int_0^\Lambda \sqrt{m^2+k^2}\,k^2 dk \approx \frac{1}{2\pi^2}\int_0^\Lambda k^3 dk = \frac{\Lambda^4}{8\pi^2} \qquad (10\text{-}22)$$

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