physical sciences forum

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*Proceedings*

# Conformal Symmetries of Strumia-Tetradis' Metric †

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† Presented at the 2nd Electronic Conference on Universe, 16 February–2 March 2023; Available online: https://ecu2023.sciforum.net/.

**Abstract:** In a recent paper [1] a new conformally flat metric was introduced, describing an expanding scalar field in a spherically symmetric geometry. The spacetime can be interpreted as a Schwarzschild-like model with an apparent horizon surrounding the curvature singularity. For the above metric, we present the **complete** conformal Lie algebra consists of a six-dimensional subalgebra of isometries (Killing Vector Fields or KVFs) and nine proper conformal vector fields (CVFs). An interesting aspect of our findings is that there exists a **gradient** (proper) conformal symmetry (i.e., its bivector $F_{ab}$ vanishes) which verifies the importance of gradient symmetries in constructing viable cosmological models. In addition, the 9-dimensional conformal algebra implies the existence of constants of motion along null geodesics that allow us to determine the complete solution of null geodesic equation.

**Keywords:** Geometric Symmetries; Conformal Vector Fields; Scalar Fields; Relativistic Cosmology

**Citation:** Apostolopoulos, P.S.; Tsipogiannis, C. Conformal Symmetries of Strumia-Tetradis' Metric. *Phys. Sci. Forum* **2023**, *3*, x. https://doi.org/10.3390/xxxxx

Published: 2 March 2023



## 1. Introduction

Conformal symmetries have been the subject of various studies during the last three decades (see e.g., [2,3]). In the majority of the cases, the main reason to investigate the existence of conformal symmetries in General Relativity was the reduction of the complexity of the resulting system of partial differential equations (pdes) in order to locate, more easily, an exact solution of the Einstein Field Equations (EFEs). However, as the generality of the underlying geometry is increased the symmetry pdes and the equation of motion become progressively highly non-linear and often lead to models without a clear physical meaning. On the other hand, there are sufficiently enough cases where physically sound models admit a conformal symmetry (proper or not) that represents an inherent constituent of their kinematical and dynamical structure. The most well known example of this situation is the isotropic, homogeneous and conformally flat Friedmann-Lemaître cosmological model which admits a 9-dimensional Lie algebra of *proper* Conformal Vector Fields (CVFs) [4]. In addition it has been shown that proper CVFs are of particular interest to construct viable astrophysical models [5–7] and at the same time it has been established the significant role of self-similar spacetimes, admitting a proper Homothetic Vector Field (HVF), since they represent the past and future (equilibrium) states for a vast number of evolving vacuum and $\gamma-$ law perfect fluid models [8,9].

Throughout this paper, the following conventions have been used: the spacetime signature is assumed $(-,+,+,+)$, lower Latin letters denote spacetime indices $a,b,...=0,1,2,3$ and we use geometrized units such that $8\pi G=c=1$.

## 2. Methods

In a recent paper [1] a new conformally flat metric was found, describing an expanding scalar bubble within a spherically symmetric geometry having a Schwarzschild-like behaviour. The solution (up to some integration constants) has the form:

$$\mathrm{C}(t,r)=\left[1-\frac{C_1^2}{\left(1+\frac{r^2-t^2}{c_2^2}\right)^2}\right]^{1/2} \tag{1}$$

where the metric of spacetime is:

$$ds^2=\mathrm{C}(t,r)^2\left[-dt^2+dr^2+r^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\right]. \tag{2}$$

The existence of a CVF $\mathbf{X}$ implies that under the infinitesimal transformation generated by $\mathbf{X}$, the spacetime metric $g_{ab}$ satisfies:

$$\mathbf{L}_{\mathrm{X}}g_{ab}=2\psi g_{ab} \tag{3}$$

where $\mathbf{L}$ is the Lie derivative along $\mathbf{X}$ and $\psi(\mathbf{X})$ denotes the conformal factor representing the scale deformation of the spacetime geometry.

The above general condition (proper CVF) specializes to a Killing Vector Field (KVF) ($\psi(\mathbf{X})=0$), to a Homothetic Vector Field (HVF) ($\psi(\mathbf{X})=$ const. $\neq 0$) and to a Special Conformal Killing Vector (SCKV) when $\psi_{;ab}=0$ (";" stands for the covariant derivative w.r.t metric $g_{ab}$).

The simplest case of a spacetime geometry admitting a maximum of 15 CVFs, is the Minkowski spacetime with metric, in Cartesian coordinates, of the form:

$$ds^2_{\mathbf{FLAT}}=-d\tau^2+dx^2+dy^2+dz^2. \tag{4}$$

The complete Lie Algebra of CVFs for the metric (4) has been determined and consists of a subalgebra of 10 KVFs, 1 proper HVF and 4 SCKVs as follows[1] [10]:

$$\mathbf{X}_1=\partial_\tau,\quad \mathbf{X}_2=\partial_x,\quad \mathbf{X}_3=\partial_y,\quad \mathbf{X}_4=\partial_z$$

$$\mathbf{X}_5=-y\partial_x+x\partial_y,\quad \mathbf{X}_6=z\partial_x-x\partial_z\quad \mathbf{X}_7=-z\partial_y+y\partial_z$$

[1]We recall that the vectors $\mathbf{X}_1-\mathbf{X}_4$ correspond to translations, $\mathbf{X}_5-\mathbf{X}_7$ to spatial rotations, $\mathbf{X}_8-\mathbf{X}_{10}$ to spacetime rotations (boosts), $\mathbf{X}_{11}$ represents the generator of the homothety and the vectors $\mathbf{X}_{12}-\mathbf{X}_{15}$ are the SCKVs.

$$\mathbf{X}_8 = x\partial_\tau + \tau\partial_x, \quad \mathbf{X}_9 = y\partial_\tau + \tau\partial_y \quad \mathbf{X}_{10} = z\partial_\tau + \tau\partial_z$$

$$\mathbf{X}_{11} = \tau\partial_\tau + x\partial_x + y\partial_y + z\partial_z$$

$$\mathbf{X}_{12} = \left(\tau^2 + x^2 + y^2 + z^2\right)\partial_\tau + 2\tau x\partial_x + 2\tau y\partial_y + 2\tau z\partial_z$$

$$\mathbf{X}_{13} = 2\tau x\partial_\tau + \left(\tau^2 + x^2 - y^2 - z^2\right)\partial_x + 2xy\partial_y + 2xz\partial_z$$

$$\mathbf{X}_{14} = 2\tau y\partial_\tau + 2xy\partial_x + \left(\tau^2 + y^2 - x^2 - z^2\right)\partial_y + 2yz\partial_z$$

$$\mathbf{X}_{15} = 2\tau z\partial_\tau + 2xz\partial_x + 2yz\partial_y + \left(\tau^2 + z^2 - x^2 - y^2\right)\partial_z.$$

For the purposes of the present work, it is convenient to transform the metric (4) and the CVFs in a form such that the geometry is foliated by spherically symmetric 2D hypersurfaces i.e., with constant and positive ($+1$) curvature. We exploit the coordinate transformation $(\tau, x, y, z) \to (t, r, \varphi, \vartheta)$:

$$\tau(t, r, \varphi, \vartheta) = t, \quad x(t, r, \varphi, \vartheta) = r\sin\varphi\sin\vartheta$$

$$y(t, r, \varphi, \vartheta) = r\cos\varphi\sin\vartheta, \quad z(t, r, \varphi, \vartheta) = r\cos\vartheta.$$

The Minkowski metric is written:

$$ds^2_{\mathbf{FLAT}} = -dt^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right)$$

whereas the CVFs take the form:

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \sin\vartheta\sin\varphi\partial_r + \frac{\cos\vartheta\sin\varphi}{r}\partial_\vartheta + \frac{\cos\varphi}{\sin\vartheta}\partial_\varphi,$$

$$\mathbf{X}_3 = \sin\vartheta\cos\varphi\partial_r + \frac{\cos\vartheta\cos\varphi}{r}\partial_\vartheta - \frac{\sin\varphi}{\sin\vartheta}\partial_\varphi$$

$$\mathbf{X}_4 = \cos\vartheta\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta$$

$$\mathbf{X}_5 = -\partial_\varphi, \quad \mathbf{X}_6 = -\cos\varphi\partial_\vartheta + \cot\vartheta\sin\varphi\partial_\varphi$$

$$\mathbf{X}_7 = \sin\varphi\partial_\vartheta + \cot\vartheta\cos\varphi\partial_\varphi$$

$$\mathbf{X}_8 = r\sin\vartheta\sin\varphi\partial_t + t\sin\vartheta\sin\varphi\partial_r + \frac{t\cos\vartheta\sin\varphi}{r}\partial_\vartheta + \frac{t\cos\varphi}{r\sin\vartheta}\partial_\varphi$$

$$\mathbf{X}_9 = r\sin\vartheta\cos\varphi\partial_t + t\sin\vartheta\cos\varphi\partial_r + \frac{t\cos\vartheta\cos\varphi}{r}\partial_\vartheta - \frac{t\sin\varphi}{r\sin\vartheta}\partial_\varphi$$

$$\mathbf{X}_{10} = r\cos\vartheta\partial_t + t\cos\vartheta\partial_r - \frac{t\sin\vartheta}{r}\partial_\vartheta$$

$$\mathbf{X}_{11} = t\partial_t + r\partial_r$$

$$\mathbf{X}_{12} = \left(r^2 + t^2\right)\partial_t + 2tr\partial_r$$

$$\mathbf{X}_{13} = 2rt\sin\vartheta\sin\varphi\partial_t + \left(r^2 + t^2\right)\sin\vartheta\sin\varphi\partial_r +$$

$$+\frac{\left(t^2 - r^2\right)\cos\vartheta\sin\varphi}{r}\partial_\vartheta + \frac{\left(t^2 - r^2\right)\cos\varphi}{r\sin\vartheta}\partial_\varphi$$

$$\mathbf{X}_{14} = 2rt\sin\vartheta\cos\varphi\partial_t + \left(r^2 + t^2\right)\sin\vartheta\cos\varphi\partial_r +$$

$$+\frac{\left(t^2 - r^2\right)\cos\vartheta\cos\varphi}{r}\partial_\vartheta + \frac{\left(r^2 - t^2\right)\sin\varphi}{r\sin\vartheta}\partial_\varphi$$

$$\mathbf{X}_{15} = 2rt\cos\vartheta\partial_t + \left(r^2 + t^2\right)\cos\vartheta\partial_r + \frac{\left(r^2 - t^2\right)\sin\vartheta}{r}\partial_\vartheta.$$

The CVFs of the conformally related metric (2) are also given by $\mathbf{X}_1 - \mathbf{X}_{15}$ with conformal factors derived from the relation:

$$\mathbf{L}_{\mathbf{X}} g_{ab} = \mathbf{L}_{\mathbf{X}}(\mathbf{C}^2\eta_{ab}) = 2[\mathbf{X}(\ln\mathbf{C}) + \Psi]g_{ab} \tag{5}$$

where $\eta_{ab}, \Psi$ is the flat metric and the conformal factors of the Minkowski spacetime respectively.

Using equation (5) we determine straightforwardly, the conformal factors of the metric (2):

$$\psi(\mathbf{X}_1) = (\ln\mathbf{C})_{,t} \qquad \psi(\mathbf{X}_2) = \sin\vartheta\sin\varphi(\ln\mathbf{C})_{,t} \qquad \psi(\mathbf{X}_3) = \sin\vartheta\cos\varphi(\ln\mathbf{C})_{,t}$$

$$\psi(\mathbf{X}_4) = \cos\vartheta(\ln \mathsf{C})_{,t} \qquad \psi(\mathbf{X}_5) = \psi(\mathbf{X}_6) = \psi(\mathbf{X}_7) = 0$$

$$\psi(\mathbf{X}_8) = \sin\vartheta\sin\varphi[t(\ln \mathsf{C})_{,r} + r(\ln \mathsf{C})_{,t}] \qquad \psi(\mathbf{X}_9) = \sin\vartheta\cos\varphi[t(\ln \mathsf{C})_{,r} + r(\ln \mathsf{C})_{,t}]$$

$$\psi(\mathbf{X}_{10}) = \cos\vartheta[t(\ln \mathsf{C})_{,r} + r(\ln \mathsf{C})_{,t}] \qquad \psi(\mathbf{X}_{11}) = 1 + t(\ln \mathsf{C})_{,r} + r(\ln \mathsf{C})_{,t}$$

$$\psi(\mathbf{X}_{12}) = 2t + 2tr(\ln \mathsf{C})_{,r} + (r^2 + t^2)(\ln \mathsf{C})_{,t}$$

$$\psi(\mathbf{X}_{13}) = \sin\vartheta\sin\varphi[2r + 2tr(\ln \mathsf{C})_{,t} + (r^2 + t^2)(\ln \mathsf{C})_{,r}]$$

$$\psi(\mathbf{X}_{14}) = \sin\vartheta\cos\varphi[2r + 2tr(\ln \mathsf{C})_{,t} + (r^2 + t^2)(\ln \mathsf{C})_{,r}]$$

$$\psi(\mathbf{X}_{15}) = \cos\vartheta[2r + 2tr(\ln \mathsf{C})_{,t} + (r^2 + t^2)(\ln \mathsf{C})_{,r}] \tag{6}$$

It is easily verified from equations (6), that the CVFs $\mathbf{X_8}, \mathbf{X_9}, \mathbf{X_{10}}$ are reduced to KVFs for the line element (2) with metric function $\mathsf{C}(t, r)$ given in (1) and represent *space-time boosts*. Note also that the 9-dimensional Lie Algebra of *proper* CVFs given above can be used, in general, to determine the general solution of the null geodesic equation. In fact the existence of a proper CVF $\mathbf{X}$ implies that there is a constant of motion along null geodesics ($n^a n_a = 0$, $n_{a;b} n^b = 0$) [6]:

$$\left(X_a n^a\right)_{;b} n^b = X_{a;b} n^a n^b + X_a n^a_{;b} n^b = \psi g_{ab} n^a n^b = 0.$$

## 3. Results and Discussion

The spacetime (1)-(2) is a solution of the field equations with a minimally coupled with gravity scalar field. It is straightforward to see that the CVF $\mathbf{X}_{11} = t\partial_t + r\partial_r$ is a **gradient** (proper) conformal symmetry (i.e., its bivector $F_{ab}$ vanishes) that verifies the importance of gradient symmetries in constructing viable cosmological models. Our findings also indicate an eventually close connection between these classes of models and the existence of a gradient CVF, so far underestimated.

**Author Contributions:** "Conceptualization, Pantelis S. Apostolopoulos; methodology, Pantelis S. Apostolopoulos; software, Pantelis S. Apostolopoulos; validation, Pantelis S. Apostolopoulos; formal analysis, Pantelis S. Apostolopoulos; Pantelis S. Apostolopoulos; resources, Pantelis S. Apostolopoulos; data curation, Pantelis S. Apostolopoulos; writing—original draft preparation, Pantelis S. Apostolopoulos; writing—review and editing, Christos Tsipogiannis; visualization, Pantelis S. Apostolopoulos; supervision, Pantelis S. Apostolopoulos; project administration, Pantelis S. Apostolopoulos. All authors have read and agreed to the published version of the manuscript."

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable here.

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