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On Closedness of Right(Left) Normal Band and Left(Right) Quasinormal Bands

1 Abstract

Abstract

It is well known that all subvarieties of the variety of all semigroups are not absolutely closed. So, it is worth to find subvarieties of the variety of all semigroups that are closed in itself or closed in the containing varieties of semigroups. We have gone through this open problem and able to determine that the varieties of right [left] normal bands and left [right] quasinormal bands are closed in the varieties of semigroups defined by the identities
$axy = xa^ny\ [axy = ay^nx],\ axy = x^nay\ [axy = ayx^n]\ (n > 1)$;
and $axy = ax^nay\ [axy = ayx^ny]\ (n > 1)$, $axy = a^nxa^ry$
$[axy = ay^rxy^n]\ (n, r \in \mathbf{N})$, respectively.

2. Introduction and Preliminaries

Introduction and Preliminaries

Let U be a subsemigroup of a semigroup S. Following Isbell [9], we say that U dominates an element d of S if for every semigroup P and for all homomorphisms $\alpha, \delta : S \longrightarrow P$ and $u\alpha = u\delta$ for every u in U implies $d\alpha = d\delta$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of semigroup S is called closed if $Dom(U, S) = U$. A semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{D}$ be a class of semigroups. A semigroup U is said to be $\mathcal{D}$-closed if $Dom(U, S) = U$ for all $S \in \mathcal{D}$ such that $U \subseteq S$. Let $\mathcal{A}$ and $\mathcal{D}$ be classes of semigroups such that $\mathcal{A}$ is a subclass of $\mathcal{D}$.

We say that $\mathcal{A}$ is $\mathcal{D}$-closed if every member of $\mathcal{A}$ is $\mathcal{D}$-closed. A class $\mathcal{D}$ of semigroups is said to be closed if $Dom(U, S) = U$ for all $U, S \in \mathcal{D}$ with U as a subsemigroup of S. Let $\mathcal{B}$ and $\mathcal{C}$ be two categories of semigroups with $\mathcal{B}$ is a subcategory of $\mathcal{C}$. It can be easily verified that a semigroup U is $\mathcal{B}$-closed if it is $\mathcal{C}$-closed.

The following result provided by Isbell [9], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Result 2.1

[9, Theorem 2.3] Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows: $d = a_0t_1 =$

$$y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \qquad (1)$$

where
$m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$,
and $a_0 = y_1a_1,\quad a_{2m-1}t_m = a_{2m},\quad a_{2i-1}t_i = a_{2i}t_{i+1},\quad y_ia_{2i} = y_{i+1}a_{2i+1}\quad (1 \leq i \leq m-1)$. Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$. We refer to the equations in Result 2.1 as the zigzag equations.

Result 2.2

[[10], Result 3] Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S \setminus U$ such that $d \in Dom(U, S)$. If (1) is a zigzag of shortest possible length m over U with value d, then $t_j, y_j \in S \setminus U$ for all $j = 1, 2, \ldots, m$.

The semigroup theoretic notations and conventions of Clifford and Preston [6] and Howie [8] will be used throughout without explicit mention.

3. Closedness of Right(Left) Normal bands

Closedness of Right(Left) Normal Bands

In general varieties of bands containing the varieties of normal bands are not absolutely closed as Higgins [7, Chapter 4] had given examples of a normal band that were not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which these varieties are closed.

As a first step in this direction, one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [11] had shown that the variety of all normal bands was closed.

Alam and Khan in [3, 4, 5] had shown that the variety of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that a variety of left [right] normal bands was closed in some containing homotypical varieties (varieties admitting an identity containing same variables on both sides) of semigroups.

In this section, we have shown that varieties of semigroups defined by the identities $axy = xa^ny\ [axy = ay^nx]$ and $axy = x^nay\ [axy = ayx^n]\ (n \in \mathbf{N})$ are closed in itself.

4. Lemma

Lemma 3.1

Let U be a subsemigroup of semigroup S such that S satisfies an identity $axy = xa^ny\ [axy = x^nay]\ (n > 1)$ and let $d \in Dom(U, S) \setminus U$ has a zigzag of type (1) in S over U with value d of shortest possible length m. Then

$$d = (\prod_{i=1}^{k} a^{n-1}_{(2i-1)})y_ka_{2k-1}t_k$$

for each $k = 1, 2, \ldots, m$.

Proof.

Let $\mathcal{V}_1 = [axy = xa^ny]$ and $\mathcal{V}_2 = [axy = x^nay]\ (n > 1)$ be the varieties of semigroups.
First we show that, in both cases whether $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$, S satisfies $xyz = y^{n-1}xyz$.

Case (i):
When $S \in \mathcal{V}_1$, for any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= yx^nz \text{ (as } S \in \mathcal{V}_1) \\ &= x^ny^nz \text{ (as } S \in \mathcal{V}_1) \\ &= y^n(x^n)^nz \text{ (as } S \in \mathcal{V}_1) \\ &= y^{n-1}(y(x^n)^nz) \\ &= (y^{n-1}x^ny)z \text{ (as } S \in \mathcal{V}_1) \\ &= (x(y^{n-1}y)z) \text{ (as } S \in \mathcal{V}_1) \\ &= y^{n-1}(yx^nz) \text{ (as } S \in \mathcal{V}_1) \\ &= y^{n-1}xyz, \qquad (2) \end{aligned}$$

as required.

Case (ii):
When $S \in \mathcal{V}_2$, then for any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= y^nxz \text{ (as } S \in \mathcal{V}_2) \\ &= x^ny^nz \text{ (as } S \in \mathcal{V}_2) \\ &= x^n(y^{n-1}yz) \\ &= (x^n(y^ny^{n-1})z) \text{ (as } S \in \mathcal{V}_2) \\ &= (y^n(y^{n-1}x)z) \text{ (as } S \in \mathcal{V}_2) \\ &= y^{n-1}xyz \text{ (as } S \in \mathcal{V}_2). \end{aligned}$$

Thus the claim is proved.

Now, we prove the lemma by using induction on k. Let U be a subsemigroup of semigroup S such that S belongs to either $\mathcal{V}_1$ or $\mathcal{V}_2$ and let $d \in Dom(U, S) \setminus U$ has a zigzag of type (1) in S over U with value d of shortest possible length m.
Now, for $k = 1$, we have

$$\begin{aligned} d &= y_1a_1t_1 \text{ (by zigzag equations)} \\ &= a_1^{n-1}y_1a_1t_1 \text{ (by equation (2)).} \end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. Then we shall show that it also holds for $k = j + 1$. Now

$$\begin{aligned} d &= (\prod_{i=1}^{j} a^{n-1}_{(2i-1)})y_ja_{2j-1}t_j \text{ (by inductive hypothesis)} \\ &= (\prod_{i=1}^{j} a^{n-1}_{(2i-1)})y_{j+1}a_{2j+1}t_{j+1} \text{ (by zigzag equations)} \\ &= (\prod_{i=1}^{j} a^{n-1}_{(2i-1)})a^{n-1}_{2j+1}y_{j+1}a_{2j+1}t_{j+1} \text{ (by equation (2))} \\ &= (\prod_{i=1}^{j+1} a^{n-1}_{(2i-1)})y_{j+1}a_{2j+1}t_{j+1}, \end{aligned}$$

as required and, by induction, the lemma is established.

5. Theorem

Theorem 3.2

The variety $\mathcal{V} = [axy = xa^ny]\ (n > 1)$ of semigroups; i.e., the class of all semigroups satisfying the identity $axy = xa^ny$ is closed.

Proof.

Take any $U, S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S) \setminus U$. Let d has a zigzag of type (1) in S over U of shortest possible length m. Now

$$d = (\prod_{i=1}^{m} a^{n-1}_{(2i-1)})y_ma_{2m-1}t_m \text{ (by Lemma 3.1)}$$

$$\begin{aligned} &= (\prod_{i=1}^{m-1} a^{n-1}_{(2i-1)})a^{n-2}_{2m-1}(a_{2m-1}(y_ma_{2m-1})t_m) \\ &\qquad \text{(for } n = 2, \text{ we treat } a^{n-2}_{2m-1}a_{2m-1} \text{ as } a_{2m-1}) \\ &= (\prod_{i=1}^{m-1} a^{n-1}_{(2i-1)})a^{n-2}_{2m-1}y_m(a_{2m-1}a^n_{2m-1}t_m) \text{ (as } S \in \mathcal{V}) \\ &= (\prod_{i=1}^{m-1} a^{n-1}_{(2i-1)})a^{n-2}_{2m-1}(y_ma_{2m-1})(a_{2m-1}t_m) \text{ (as } S \in \mathcal{V}) \\ &= (\prod_{i=1}^{m-1} a^{n-1}_{(2i-1)})a^{n-2}_{2m-1}y_{m-1}a_{2m-2}a_{2m} \text{ (by zigzag equations)} \end{aligned}$$

Theorem 3.2

The variety $\mathcal{V} = [axy = xa^ny]\ (n > 1)$ of semigroups; i.e., the class of all semigroups satisfying the identity $axy = xa^ny$ is closed.

Proof.

Take any $U, S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S) \setminus U$. Let d has a zigzag of type (1) in S over U of shortest possible length m. Now

$$d = (\prod_{i=1}^{m} a^{n-1}_{(2i-1)})y_ma_{2m-1}t_m \text{ (by Lemma 3.1)}$$

$$\vdots$$

$$\begin{aligned} &= a_1^{n-1}a_3^{n-2}\cdots a^{n-2}_{2m-3}a^{n-2}_{2m-1}y_1a_2a_4\cdots a_{2m-2}a_{2m} \\ &= a_1^{n-2}(a_1(a_3^{n-2}\cdots a^{n-2}_{2m-3}a^{n-2}_{2m-1}y_1a_2)(a_4\cdots a_{2m-2}a_{2m})) \\ &= a_1^{n-2}a_3^{n-2}\cdots a^{n-2}_{2m-3}a^{n-2}_{2m-1}y_1(a_2a_1^na_4)a_6\cdots a_{2m-2}a_{2m} \text{ (as } S \in \mathcal{V}) \\ &= a_1^{n-2}a_3^{n-2}\cdots a^{n-2}_{2m-3}a^{n-2}_{2m-1}(y_1a_1)a_2a_4a_6\cdots a_{2m-2}a_{2m} \text{ (as } U \in \mathcal{V}) \\ &= a_1^{n-2}a_3^{n-2}\cdots a^{n-2}_{2m-3}a^{n-2}_{2m-1}a_0a_2a_4\cdots a_{2m-2}a_{2m} \\ &\qquad \text{(by zigzag equations)} \end{aligned}$$

$\in U$

$\Rightarrow Dom(U, S) = U$.
Thus the proof of the theorem is complete.

The following corollary is an immediate consequence of Theorem 3.2:

Corollary 3.3

The variety of all right normal bands is closed in the variety $\mathcal{V} = [axy = xa^ny]\ (n > 1)$ of semigroups.

Dually, we may prove the following result.

Theorem 3.4

The variety $\mathcal{V} = [axy = ay^nx]\ (n > 1)$ of semigroups; i.e., the class of all semigroups satisfying the identity $axy = ay^nx$ is closed.

6. Closedness of Left(Right) Quasinormal Bands

Closedness of Left(Right) Quasinormal Bands

In this section, we have shown that varieties of semigroups defined by the identities $axy = ax^nay\ [axy = ayx^ny]\ (n > 1)$ and $axy = a^nxa^ry\ [axy = ay^rxy^n]\ (n, r \in \mathbf{N})$, are closed in itself.

Theorem 4.1

The variety $\mathcal{V} = [axy = ax^nay]\ (n > 1)$ of semigroups; i.e., the class of all semigroups satisfying the identity $axy = ax^nay$ is closed.

Proof.

Take any $U, S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S) \setminus U$. Let d has zigzag of type (1) in S over U of shortest possible length m. In order to prove the theorem, we first prove the following lemma. □

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Lemma 4.2

$$y_ia_{2i-1}t_i = y_ia^n_{2i-1}y_{i+1}a_{2i+1}t_{i+1}\ (1 \leq i \leq m-1).$$

Now,

$$\begin{aligned} d &= a_0t_1 \text{ (by zigzag equations)} \\ &= y_1a_1t_1 \text{ (by zigzag equations)} \\ &= y_1a_1^n(y_2a_3t_2) \text{ (by Lemma 4.2)} \\ &= y_1a_1^ny_2a_3^ny_3a_5t_3 \text{ (by Lemma 4.2)} \\ &\ \ \vdots \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}(y_{m-1}a_{2m-3}t_{m-1}) \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a^n_{2m-3}(y_ma_{2m-1}t_m) \text{ (by Lemma 4.2)} \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a^n_{2m-3}y_ma^n_{2m-1}y_mt_m \text{ (as } S \in \mathcal{V}) \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a^n_{2m-3}y_ma^{n-2}_{2m-1}(a_{2m-1}a_{2m-1}y_m)t_m \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a^n_{2m-3}y_ma^{n-2}_{2m-1}a_{2m-1}a^n_{2m-1}a_{2m-1}y_mt_m \\ &\qquad \text{(as } S \in \mathcal{V}) \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a^n_{2m-3}(y_m(a_{2m-1}a_{2m-1})^ny_mt_m) \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a^n_{2m-3}(y_ma_{2m-1})(a_{2m-1}t_m) \text{ (as } S \in \mathcal{V}) \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}(y_{m-1}a^n_{2m-3}y_{m-1}a_{2m-2})a_{2m} \\ &\qquad \text{(by zigzag equations)} \\ &= y_1a_1^ny_2a_3^n\cdots y_{m-2}a^n_{2m-5}y_{m-1}a_{2m-3}a_{2m-2}a_{2m} \text{ (as } S \in \mathcal{V}) \\ &\ \ \vdots \\ &= y_1a_1^n(y_2a_3)a_4a_6\cdots a_{2m-2}a_{2m} \\ &= (y_1a_1^ny_1a_2)a_4a_6\cdots a_{2m-2}a_{2m} \text{ (by zigzag equations)} \\ &= y_1a_1a_2a_4a_6\cdots a_{2m-2}a_{2m} \text{ (as } S \in \mathcal{V}) \\ &= a_0a_2a_4a_6\cdots a_{2m-2}a_{2m} \text{ (by zigzag equations)} \end{aligned}$$

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$\in U$

$\Rightarrow Dom(U, S) = U$.
Thus the proof of the theorem is complete. The following corollaries are immediate consequences of the Theorem 4.1.

Corollary 4.3

The variety of all left quasinormal bands is closed in the variety $\mathcal{V} = [axy = ax^nay]\ (n > 1)$ of semigroups.

Dually, we have the following result.

Theorem 4.4

The variety $\mathcal{V} = [axy = ayx^ny]\ (n > 1)$ of semigroups; i.e., the class of all semigroups satisfying the identity $axy = ayx^ny$ is closed.

Aligarh Muslim University, India

7. Open Problem

Open Problem

1. Are the varieties considered in the paper absolutely closed; if not, of finding out largest varieties of semigroups in which these varieties are closed.

8. References

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Shabnam Abbas (Research Scholar)
Departmentof Mathematics, AMU Aligarh-202002, India
shabnamabbas.25@gmail.com

Dr. Wajih Ashraf (Assistant Professor)
Departmentof Mathematics, AMU Aligarh-202002, India
wajih.mm@amu.ac.in