computer sciences and mathematics forum

Proceeding Paper

Generalization and Sharpening of Some Inequalities for Polynomials †

Ishfaq Nazir *, Irfan Ahmad Wani and Mohammad Ibrahim Mir

Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India; email1@email.com (I.A.W.); ibrahimmath80@gmail.com (M.I.M.)
* Correspondence: ishfaqnazir02@gmail.com

† This paper is an extended version of our paper published in Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.

Abstract: This paper's objective is to develop some findings for the polar derivative of a polynomial in the plane that are motivated by a classical result of Turán that connects the sup-norm of the derivative on the unit circle to that of the polynomial itself under some conditions. The obtained results strengthen and broaden certain existing estimates that relate the sup-norm of the polar derivative and the polynomial. In addition, a few numerical examples are provided to demonstrate how, in some cases, the bounds produced by our findings may be far sharper than those previously discovered in the extensive literature on this topic.

Keywords: polynomials; inequalities; zeros

MSC: Primary 30A10; 30C10; Secondary 30D15

check for updates

Citation: Nazir, I.; Wani, I.A.; Mir, M.I. Generalization and Sharpening of Some Inequalities for Polynomials. *Comput. Sci. Math. Forum* **2023**, *1*, 0. https://doi.org/

Academic Editor: Firstname Lastname

Published: 28 April 2023

1. Introduction

A classical analytic subject is the study of extremal problems of functions and the conclusions where certain techniques to obtaining polynomial inequalities for various norms and with varied constraints on utilising different methods of the geometric function theory. The literature for proving the inverse theorems in approximation theory heavily relies on the Erdós-Lax and Turán-type inequalities connecting the norm of the derivative and the polynomial itself as well as generalising the classical polynomial inequalities. Of course, these inequalities also have their own intrinsic interest. As evidenced by numerous recent studies, these inequalities for constrained polynomials have been the focus of numerous research works (for example, see [4–6,8–10]).

According to well known inequality of Bernstein [2] on the derivative of a polynomial $P(z)$ of degree n, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

The result is best possible and equality holds for a polynomial having all its zeros at the origin.

Erdös conjectured and later Lax [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{2}$$

The inequality (2) is best possible and equality holds for $P(z) = a + bz^n$, where $|a| = |b|$. As an extension of (2), Malik [10] proved that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)| \tag{3}$$

For the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise estimate of maximum $|P'(z)|$ on $|z| = 1$ is not easily obtainable. For quite some time it was believed that if $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, then the inequality analogous to (3) should be

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)| \tag{4}$$

till Professor Saff gave the example $P(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief. In 1980, it was shown by Govil [4] that (4) holds with an additional hypothesis and proved the following result.

Theorem 1. *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)| \tag{5}$$

The result is best possible and equality holds in (5) for $P(z) = z^n + k^n$.

Singh and Chanam [14] have proved the following refinement of inequality (5) by using the Dubinin lemma [2].

Theorem 2. *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{1}{1+k^n} \left(n - \frac{(\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n|})k^n}{\sqrt{|c_0|}} \right) \max_{|z|=1} |P(z)| \tag{6}$$

The result is best possible and equality holds in (6) for $P(z) = z^n + k^n$.

On the other hand, P. Turán [12] proved in 1939 that if a polynomial $P(z)$ of degree n has all of its zeros in $|z| \leq 1$, then it has a lower bound estimate of the derivative to the size of the polynomial.

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{7}$$

Also using lemma of Dubinin [3], Singh and Chanam [14] have proved the following refinement of Turán's inequality.

Theorem 3. *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n} \left(n + \frac{k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0|}}{k^{n/2}\sqrt{|c_n|}} \right) \max_{|z|=1} |P(z)| \tag{8}$$

The result is best possible and equality holds in (8) for $P(z) = z^n + k^n$.

Recently Authors [11] have obtained following refinement of Theorem 2.

Theorem 4. *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leq t \leq 1$, we have*

$$\max_{|z|=1} |P'(z)| \leq \left[n - \frac{k^n}{1+k^n} \left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n| + tm}}{\sqrt{|c_0|}} \right) \right] \max_{|z|=1} |P(z)| - \frac{k^n}{1+k^n} \left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n| + tm}}{\sqrt{|c_0|}} \right) tm \tag{9}$$

where $m = \min_{|z|=1/k} |Q(z)|$. Equality holds in (9) for $P(z) = z^n + k^n$.

In the same paper, Authors [11] have also obtained following refinement of Theorem 3.

Theorem 5. *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k$, $k \geq 1$, then for $0 \leq t \leq 1$, we have*

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n} \left(n + \frac{k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0| + tm}}{k^{n/2}\sqrt{|c_n|}} \right) \left(\max_{|z|=1} |P(z)| + tm \right) \tag{10}$$

where $m = \min_{|z|=k} |P(z)|$. Equality holds in (2.1) for $P(z) = z^n + k^n$.

Different versions of these Bernstein and Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. The one such generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative which is defined as

Definition: Let $p(z)$ be a polynomial of degree n with complex coefficients and $\alpha \in \mathbb{C}$ be a complex number, then the polynomial

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$$

is called polar derivative of $p(z)$ with pole α. Note that $D_\alpha p(z)$ is a polynomial of degree $n-1$ and it is a generalisation of the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Many of the generalizations of above mentioned inequalities involve the comparison of the polar derivative $D_\alpha P(z)$ with various choices of $p(z)$, α and other parameters. For more information on the polar derivative of polynomials one can consult the comprehensive books of Marden [8], Milovanonic et al. [9] or Rahman and Schmeisser [13]. In 1998, Aziz and Rather [1] extended inequality () to polar derivative by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k}{1+k^n} \right) \max_{|z|=1} |P(z)| \tag{11}$$

Govil and Mctume [6] established the polar derivative extension of inequality (11) and proved

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k}{1+k^n} \right) \max_{|z|=1} |P(z)| + n \left(\frac{|\alpha| - (1+k+k^n)}{1+k^n} \right) \min_{|z|=k} |P(z)| \tag{12}$$

for any complex number α with $|\alpha| \geq 1 + k + k^n$.

Also, using Dubinin lemma [3], Singh and Chanam [14] have proved the following improvement of inequality (11) due to Aziz and Rather [1].

Theorem 6. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k$, $k \geq 1$, then for any complex number α with $|\alpha| \geq k$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{(1+k^n)} \left(n + \frac{k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0|}}{k^{n/2}\sqrt{|c_n|}} \right) \max_{|z|=1} |P(z)| \tag{13}$$

As a polar derivative generalization of Theorem 1, Mir and D. Breaz [12] obtained following result.

Theorem 7. *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{|\alpha| + k^n}{1+k^n} \right) \max_{|z|=1} |P(z)| \tag{14}$$

The result is best possible and equality holds in (5) for $P(z) = z^n + k^n$.

2. Main Results

We begin, by presenting the following generalization and refinement of inequality (1), (2) and Theorem 6.

Theorem 8. *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k$, $k \geq 1$, then for any complex number α with $|\alpha| \geq 1 + k + k^n$, and $0 \leq t \leq 1$, we have*

$$\begin{aligned}
\max_{|z|=1} |D_\alpha P(z)| \geq{}& \frac{(|\alpha| - k)}{(1+k^n)} \left(n + \frac{k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0| + tm}}{k^{n/2}\sqrt{|c_n|}} \right) \max_{|z|=1} |P(z)| \\
&+ t \left(\frac{n(|\alpha| - (1+k+k^n))}{(1+k^n)} + \frac{(|\alpha| - k)}{(1+k^n)} \frac{(k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0| + tm})}{k^{n/2}\sqrt{|c_n|}} \right) m
\end{aligned} \tag{15}$$

where $m = \min_{|z|=k} |P(z)|$.

Remark 1. *Since $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ has all its zeros in the disk $|z| \leq k$, $k \geq 1$ and if $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then*

$$\left| \frac{c_0}{c_n} \right| = |z_1 z_2 \ldots z_n| = |z_1||z_2| \ldots |z_n| \leq k^n$$

As we see in the proof of Theorem 8 (given in next section), for every β with $|\beta| \leq 1$, the polynomial $P(z) + \beta m$ has all its zeros in the disk $|z| \leq k$, $k \geq 1$, hence

$$\left| \frac{c_0 + \beta m}{c_n} \right| \leq k^n \tag{16}$$

which is equivalent to

$$k^{n/2}\sqrt{|c_n|} \geq \sqrt{|c_0 + \beta m|}$$

If in (16), we choose the argument of β suitably, we get

$$\sqrt{|c_0| + |\beta| m} \leq k^{n/2}\sqrt{|c_n|}. \tag{17}$$

If we take $|\beta| = t$ *in (17), so that* $0 \le t \le 1$*, we get* $\sqrt{|c_0| + tm} \le k^{n/2}\sqrt{|c_n|}$.

Remark 2. *For* $t = 0$*, Theorem 8 reduces to Theorem 6.*

Remark 3. *If we divide (15) by* $|\alpha|$ *and take* $|\alpha| \to \infty$*, we get Theorem 5 and thus Theorem 8 contains Theorem 5.*

Remark 4. *If we divide (15) by* $|\alpha|$ *and take* $|\alpha| \to \infty$ *and* $t = 0$ *we get Theorem 3 and thus Theorem 8 also contains Theorem 3.*

Theorem 8 in general provides much better information regarding $\max_{|z|=1}|D_\alpha P(z)|$, in case when $P(z)$ has all its zeros in $|z| < k$, $k \ge 1$. We illustrate this with the help of following example.

Example 1. *Consider* $P(z) = z^2 + 3z + 5/4$*, which is polynomial of degree* 2 *having all its zeros in* $|z| \le 5/2$*. We take* $k = 3$ *and* $\alpha = 15 + 8i$*, so that* $|\alpha| = 17$*, then clearly* $|\alpha| \ge 1 + k + k^n$*. We find that*

$$\min_{|z|=3}|P(z)| = 5/4 \quad and \quad \max_{|z|=1}|P(z)| = 21/4$$

For this polynomial, we obtain that

$$\max_{|z|=1}|D_\alpha P(z)| \ge 14.70 \quad (by \quad inequality \quad (1.1))$$

$$\max_{|z|=1}|D_\alpha P(z)| \ge 18.20 \quad (by \quad inequality \quad (1.12))$$

$$\max_{|z|=1}|D_\alpha P(z)| \ge 19.55 \quad (by \quad Theorem \quad 1.6)$$

While Theorem 8 (with t = 1), gives

$$\max_{|z|=1}|D_\alpha P(z)| \ge 20.25$$

which is much better than the bound given by above estimates.

Using Theorem 8, we prove the following generalisation and refinement of Theorem 4 and Theorem 7.

Theorem 9. *Let* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ *be a polynomial of degree n having no zero in* $|z| < k$, $k \le 1$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$*. If* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on* $|z| = 1$*, then for* $0 \le t \le 1$*, we have*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \le & \left[n|\alpha| - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{k^{n/2}\sqrt{|c_0|} - \sqrt{|c_n| + tm}}{\sqrt{|c_0|}} \right) \right] \max_{|z|=1}|P(z)| \\ & - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0| + tm}}{\sqrt{|c_0|}} \right) tm \end{aligned} \tag{18}$$

where $m = \min_{|z|=1/k}|Q(z)|$*. Equality holds in (18) for* $P(z) = z^n + k^n$*.*

Remark 5. *If we divide (18) by* $|\alpha|$ *and take* $|\alpha| \to \infty$*, we get Theorem 4 and thus Theorem 9 contains Theorem 4.*

Taking $t = 0$ in Theorem 9, we get the following polar derivative generalization of Theorem 7.

Corollary 1. *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leq t \leq 1$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left[n|\alpha| - \frac{(|\alpha|-1)k^n}{(1+k^n)} \left(n + \frac{k^{n/2}\sqrt{|c_0|} - \sqrt{|c_n|}}{\sqrt{|c_0|}} \right) \right] \max_{|z|=1} |P(z)| \tag{19}$$

where $m = \min_{|z|=1/k} |Q(z)|$. Equality holds in (18) for $P(z) = z^n + k^n$.

Remark 6. *If we divide (18) by $|\alpha|$ and take $|\alpha| \to \infty$ and $t = 0$ we get Theorem 2.*

Remark 7. *If we divide (18) by $|\alpha|$ and take $|\alpha| \to \infty$, $t = 0$ and $k = 1$ we get the following improvement of inequality (2) due to Erdös and Lax for a subclass of polynomials having all its zeros in $|z| \geq 1$.*

Corollary 2. *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{1}{2} \left(n - \frac{\sqrt{|c_0|} - \sqrt{|c_n|}}{\sqrt{|c_0|}} \right) \max_{|z|=1} |P(z)| \tag{20}$$

In the same way, Theorem 9 in general provides much better information than Theorem regarding the maximum of $|D_\alpha P(z)|$ on $|z| = 1$. We illustrate this with the help of following example

Example 2. *Consider $P(z) = z^3 - z^2 + z - 1$, which is polynomial of degree 3. Clearly, $P(z)$ has all its zeros in $|z| \leq 1$. Further*

$$Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)} = -P(z).$$

So that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. We take $k = \frac{1}{2}$, so that $P(z) \neq 0$ in $|z| < k = \frac{1}{2}$ and we find numerically that $\max_{|z|=1} |P(z)| = 4$, $\min_{|z|=\frac{1}{1/2}} |Q(z)| = \min_{|z|=2} |Q(z)| = 5$. Taking $\alpha = \frac{3+i\sqrt{7}}{2}$, so that $|\alpha| = 2$, we obtain the following estimates

$$\max_{|z|=1} |D_\alpha P(z)| \leq 22.66 \quad (by \quad (14))$$

While Theorem 9 gives

$$\max_{|z|=1} |D_\alpha P(z)| \leq 20.85$$

which is much better than the bound given by (14).

For the proof our results, we need the following lemma due to Govil and Rahman [4].

Lemma 1. *If $P(z)$ is a polynomial of degree n then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

3. Proofs of Theorems

Proof of Theorem 8. If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ has a zero on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and the result follows from Theorem 3 in this case. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k$, $k \geq 1$.

Let $H(z) = P(kz)$ and $G(z) = z^n\overline{H(1/\overline{z})} = z^n\overline{P(k/\overline{z})}$, then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$. This gives

$$\left|z^n\overline{P\left(\frac{k}{\overline{z}}\right)}\right| = |P(kz)| \geq m \quad for \quad |z| = 1$$

It follows by Minimum Modulus principle, that

$$\left|z^n\overline{P\left(\frac{k}{\overline{z}}\right)}\right| \geq m \quad for \quad |z| \leq 1$$

Replacing z by $1/\overline{z}$, it implies that

$$|P(kz)| \geq m|z|^n \quad for \quad |z| \geq 1$$

or

$$|P(z)| \geq m\left|\frac{z}{k}\right|^n \quad for \quad |z| \geq k \tag{21}$$

Now, consider the polynomial

$$F(z) = P(z) + \beta m$$

where β is complex number with $|\beta| \leq 1$, then all the zeros of $F(z)$ lie in $|z| \leq k$. Because, if for some $z = z_1$, with $|z_1| > k$, we have

$$F(z_1) = P(z_1) + \beta m = 0$$

then

$$|P(z_1)| = |\beta m| \leq m < m\left|\frac{z_1}{k}\right|^n$$

which contradicts (21). Hence, for every complex number β with $|\beta| \leq 1$, the polynomial

$$F(z) = P(z) + \beta m = (c_0 + \beta m) + \sum_{\nu=1}^{n} c_\nu z^\nu$$

has all its zeros in $|z| \leq k$, where $k \geq 1$. Applying Theorem 6, to the polynomial $F(z)$, we get for every complex number β with $|\beta| \leq 1$ and $|z| = 1$

$$\max_{|z|=1}|D_\alpha(P(z) + \beta m)| \geq \frac{|\alpha| - k}{1 + k^n}\left(n + \frac{k^{n/2}\sqrt{|c_n|} - \sqrt{|c_0 + \beta m|}}{k^{n/2}\sqrt{|c_n|}}\right)\left(\max_{|z|=1}|P(z) + \beta m|\right) \tag{22}$$

For every $\beta \in \mathbb{C}$, we have

$$|c_0 + \beta m| \leq |c_0| + |\beta|m,$$

Since the function $k(x) = n + \frac{k^{n/2}\sqrt{|c_n|}-\sqrt{x}}{k^{n/2}\sqrt{|c_n|}}$ is decreasing for $k \geq 1$, it follows from (22) that for every β with $|\beta| \leq 1$ and $|z| = 1$

$$\max_{|z|=1} |D_\alpha(P(z)+\beta m)| \geq \frac{|\alpha|-k}{1+k^n}\left(n + \frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+|\beta|m}}{k^{n/2}\sqrt{|c_n|}}\right)\left(\max_{|z|=1}|P(z)+\beta m|\right) \tag{23}$$

Choosing argument of β on R.H.S of (23) such that

$$\max_{|z|=1}|P(z)+\beta m| = \max_{|z|=1}|P(z)| + |\beta| m,$$

we obtain from (23) that

$$\max_{|z|=1} |D_\alpha P(z)| + |\beta| mn \geq \frac{|\alpha|-k}{1+k^n}\left(n + \frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+|\beta|m}}{k^{n/2}\sqrt{|c_n|}}\right)\left(\max_{|z|=1}|P(z)| + |\beta| m\right) \tag{24}$$

which on taking $|\beta| = t$, so that $0 \leq t \leq 1$ gives

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \geq{}& \frac{(|\alpha|-k)}{(1+k^n)}\left(n + \frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}}{k^{n/2}\sqrt{|c_n|}}\right)\max_{|z|=1}|P(z)| \\ &+ t\left(\frac{n(|\alpha|-(1+k+k^n))}{(1+k^n)} + \frac{(|\alpha|-k)}{(1+k^n)}\frac{(k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm})}{k^{n/2}\sqrt{|c_n|}}\right)m \end{aligned} \tag{25}$$

This completes the proof of the Theorem 8.
□

Proof of Theorem 9. Note that for any complex number α with $|\alpha| \geq 1$, we have on $|z| = 1$

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &= |nP(z) - zP'(z) + \alpha P'(z)| \\ &\leq |nP(z) - zP'(z)| + |\alpha||P'(z)| \\ &= |Q'(z)| + |\alpha||P'(z)| \\ &= n\max_{|z|=1}|P(z)| - |P'(z)| + |\alpha||P'(z)| \quad (by \quad Lemma 1) \\ &= n\max_{|z|=1}|P(z)| + (|\alpha|-1)|P'(z)| \end{aligned}$$

Therefore using Theorem 4, we have

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \leq n\max_{|z|=1}|P(z)| + (|\alpha|-1)&\left[n - \frac{k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|}-k^{n/2}\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)| \\ - \frac{(|\alpha|-1)k^n}{(1+k^n)}&\left(n + \frac{\sqrt{|c_0|}-k^{n/2}\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)tm \end{aligned}$$

That is

$$\max_{|z|=1}|D_\alpha P(z)| \leq \left[n|\alpha| - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)| - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)tm \tag{26}$$

□

Author Contributions:

Funding:

Institutional Review Board Statement:

Informed Consent Statement:

Data Availability Statement: All data generated or analysed during this study are included in this article.

Conflicts of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Aziz, A.; Rather, N.A. refinement of a theorem of Paul Turán concerning polynomials. *Math. Inequal. Appl.* **1998**, *1*, 231–238.
2. Bernstein, S.N. *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*; Gauthier-Villars: Paris, France, 1926.
3. Dubinin, V.N. Distortion theorems for polynomials on the unit circle. *Russ. Acad. Sci. Sb. Math.* **2000**, *191*, 1797–1807.
4. Govil, N.K.; Rahman, Q.I. Functions of exponential type not vanishing in a half-plane and related polynomials. *Trans. Am. Math. Soc.* **1969**, *137*, 501–517.
5. Govil, N.K. On a theorem of Bernstein. *Proc. Natl. Acad. Soc. India* **1980**, *50*, 50–52.
6. Govil, N.K.; Mctume, G.N. Some generalizations involving the polar derivative for an inequality of Paul Turán. *Acta Math. Hungar.* **2004**, *104*, 115–126.
7. Lax, P.D. Proof of a conjecture of P. Erdös on the derivative of a polynomial. *Bull. Am. Math. Soc.* **1944**, *50*, 509–513.
8. Marden, M. *Geometry of Polynomials*; American Mathematical Society: Providence, RI, USA, 1966; Volume 3.
9. Milovanović, G.V.; Mitrinović, D.S.; Rassias, T.M. *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*; World Scientific Publishing Co.: Singapore, 1994.
10. Malik, M. On the derivative of a polynomial. *J. Lond. Math. Soc.* **1969**, *1*, 57–60.
11. Mir, M.I.; Nazir, I.; Wani, I.A. On Erdös-Lax and Turán-type Inequalities for Polynomials. *Asian-Eur. J. Math.* **2023**, *16*, 2350039.
12. Mir, A.; Breaz, D. Bernstein and Turán-type inequalities for a polynomial with constraints on its zeros. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math.* **2021**, *115*, 124.
13. Rahman, Q.I.; Schmeisser, G. *Analytic Theory of Polynomials*; Oxford University Press: Oxford, UK, 2002.
14. Singh, T.B.; Chanam, B. Generalization and sharpening of certain Bernstein and Turán-type inequalities for the polar derivative of a Polynomial. *J. Math. Inequal.* **2021**, *15*, 1663–1675.
15. Turán, P. Über die Ableitung von Polynomen. *Compos. Math.* **1939**, *7*, 89–95.