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Global Well-Posedness for 3d-Axisymmetric Anisotropic Boussinesq System with Stratification Effects †

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Abstract: In this work, we study the Cauchy problem of the 3d-Anisotropic Boussinesq system with Horizontal dissipation and stratification effect for axisymmetric data. This system couples the Navier-stokes equation with a non-homogenous transport-diffusion equation. More precisely, we prove that this kind of system admits a unique global solution belongs to the Anisotropic Sobolev spaces.

Keywords: Anisotropic Sobolev spaces; boussinesq system; global solutions; uniqueness

1. Introduction

The $3d$-Boussinesq system with horizental dissipation and stratification effects describes the convective motion of a viscous or inviscid fluid, which reads as follows:

$$\begin{cases} \partial_t v + v \cdot \nabla v - \upsilon_h(\partial^2_{x_1} v + \partial^2_{x_2} v) + \nabla p = \varrho \vec{e_3} & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \varrho + v \cdot \nabla \varrho - \kappa_h(\partial^2_{x_1} \varrho + \partial^2_{x_2} \varrho) = -\mathcal{N}^2(x_3) v_3 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \text{div} v = 0, \\ (v, \varrho)_{|t=0} = (v_0, \varrho_0). \end{cases} \qquad (B_{\upsilon_h,\kappa_h})$$

where $v = (v_1, v_2, v_3)$ describes the velocity, ϱ is a scalar function represents the density, p is the pressure and υ_h, κ_h denotes viscosity and thermal diffusivity of the fluid. On the other hand it deserves to recall that the action of the buoyant forces represented by ϱe_3, where $e_3 = (0,0,1)^t$ induce the motion derived from the gravitational potential energy and the quantity $\mathcal{N}^2$ is a scalar function called the Brunt–Väisärä frequency or the stratification parameter (see,e.g, [1,2]). One of the interesting situations is to investigate the global existence of the system $(B_{\upsilon_h,\kappa_h})$ is the case when the velocity is axisymmetric without swirl, which corresponds to the following case: we say that the velocity v is axisymetric without swirl if we have:

$$v(t, x_1, x_2, x_3) \triangleq v_r(t,r,z)\vec{e_r} + v_z(t,r,z)\vec{e_z},$$

with $x_1 = r\cos\theta, \quad x_2 = r\sin\theta$ and $x_3 = z$, where $r = \sqrt{x_1^2 + x_2^2}, \quad 0 \le \theta \le 2\pi$. The triplet (e_r, e_θ, z) is the cylindrical basis given by

$$\vec{e_r} = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad \vec{e_\theta} = (-\sin\theta, \cos\theta, 0) \quad \text{and} \quad \vec{e_z} = (0,0,1).$$

Lately, there are many papers which study the $3d$-axisymmetric Boussinesq system without swirl for different viscosities, we refer the reader to [3–6], see also [7–10] for the axisymetric Euler equation. In 2012, Miao and Zheng investigated the system $(B_{\upsilon_h,\kappa_h})$ with axisymetric data but without stratification effect i.e. $(\mathcal{N} \equiv 0)$, where they have proved the existence and uniqueness of solutions. In this paper we aim to extend the result of [5] to the Boussinesq system with stratification effect, more precisely, the main result of this paper is stated as follows:

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2. Main Result

Theorem 1. *Let $v_0 \in H^1$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2, \partial_z\omega_0 \in L^2$ and let $\varrho_0 \in H^{0,1}$ an axisymmetric function. Then the system (B_{v_h,κ_h}) admits a unique global solution such that*

$$(v,\varrho) \in \big(\mathcal{C}(\mathbb{R}_+;H^1) \cap L^2_{loc}(\mathbb{R}_+;H^{1,2} \cap H^{2,1}) \cap L^1_{loc}(\mathbb{R}_+;Lip)\big) \times \mathcal{C}(\mathbb{R}_+;H^{0,1} \cap H^{1,1}).$$

Moreover, we have

$$\frac{\omega}{r} \in L^\infty_{loc}(\mathbb{R}_+;L^2) \cap L^2_{loc}(\mathbb{R}_+;H^{1,0}), \quad \partial_z\omega \in \mathcal{C}(\mathbb{R}_+;L^2) \cap L^2_{loc}(\mathbb{R}_+;H^{1,0}).$$

Remark 1. *Without loss of generality, we assume that $\mathcal{N}^2(x_3) = v_h = \kappa_h = 1$.*

In the spirit of [4,5], by using the hide structure between the velocity equation and the density equation in the system (B_{v_h,κ_h}), we can introduce a new quantity which will be called by the coupled function of the system (B_{v_h,κ_h}), this quantity solves the following equation

$$\partial_t\widetilde{\Gamma} + v\cdot\nabla\widetilde{\Gamma} - \Delta_h\widetilde{\Gamma} + \frac{2}{r}\partial_r\widetilde{\Gamma} = -\frac{1}{2}v_z,$$

with $\Delta_h \triangleq \partial^2_{x_1} + \partial^2_{x_2}$ and $\widetilde{\Gamma} \triangleq \zeta - \frac{\varrho}{2}$.

From this observation we start by controlling the L^2-norm of $\widetilde{\Gamma}$, hence we bound the following quantity $\|\frac{\omega_\theta}{r}\|_{L^2}$, then we get the global H^1-estimate of the velocity, by combining these results, we obtain the Lipschitz norm of the velocity, which leads us to the global existence os solutions.

3. Proof of Theorem 1

Proposition 1. *Let (v,ϱ) be a smooth solution of (B_{v_h,κ_h}). Then we have*

$$\|v(t)\|^2_{L^2} + \|\varrho(t)\|^2_{L^2} + \int_0^t (\|\nabla_h\varrho(\tau)\|^2_{L^2} + \|\nabla_h v(\tau)\|^2_{L^2})d\tau \le C_0e^{C_0t} \tag{1}$$

Proposition 2. *Let (ω,ϱ) be a solution for (B_{v_h,κ_h}) and $\partial_z\varrho_0, \partial_z\omega_0 \in L^2$. Then*

$$\|\partial_z\varrho(t)\|_{L^2} + \int_0^t \|\nabla_h\partial_z\varrho(\tau)\|_{L^2}d\tau \le C_0e^{exp(C_0t)}, \tag{2}$$

$$\|\partial_z\omega(t)\|_{L^2} + \int_0^t \|\nabla_h\partial_z\omega(\tau)\|_{L^2}d\tau \le C_0e^{exp(C_0t)}. \tag{3}$$

Proof. We set $\sigma(t,x) = \partial_z\varrho(t,x)$ and by applying ∂_z to the density equation in (B_{v_h,κ_h}) we find

$$\partial_t\sigma + v\cdot\nabla\sigma - \Delta_h\sigma = \partial_z v_z(1-\sigma(t,x)) - \partial_z v_r\partial_r\varrho,$$

On the other hand from incompresiblity condition, we have $\partial_z v_z = -\partial_r v_r - \frac{v_r}{r}$. Then the classical L^2-estimates gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sigma(t)\|^2_{L^2} + \|\nabla_h\sigma(t)\|^2_{L^2} &= -\int_{\mathbb{R}^3}\partial_z v_r\partial_r\varrho\sigma dx + \int_{\mathbb{R}^3}\partial_r v_r\sigma^2dx + \int_{\mathbb{R}^3}\frac{v_r}{r}\sigma^2dx \\ &+ \int_{\mathbb{R}^3}\partial_z v_z\sigma dx \\ &= \sum_{i=1}^4 \mathrm{I}_i.\end{aligned} \tag{4}$$

We start by I_1, we use the anisotropic inequality [5, Lemma F.3] and Hölder inequality, we obtain

$$
\begin{aligned}
(\mathrm{I}_1)^2 &\lesssim \|\partial_z v_r\|_{L^2}\|\nabla_h\partial_z v_r\|_{L^2}\|\partial_r\varrho\|_{L^2}\|\partial_z\partial_r\varrho\|_{L^2}\|\sigma\|_{L^2}\|\nabla_h\sigma\|_{L^2}\\
&\lesssim \|\partial_z v_r\|_{L^2}\|\nabla_h\partial_z v_r\|_{L^2}\|\partial_r\varrho\|_{L^2}\|\sigma\|_{L^2}\|\nabla_h\sigma\|^2_{L^2}.
\end{aligned}
$$

Young's inequality and Biot savart law ensures that

$$
\mathrm{I}_1 \lesssim \|\nabla_h\partial_z v_r\|^2_{L^2} + \|\omega\|^2_{L^2}\|\partial_r\varrho\|^2_{L^2}\|\sigma\|^2_{L^2} + \frac{1}{4}\|\nabla_h\sigma\|^2_{L^2}.
$$

For I_2, from the anisotropic inequality [5, Lemma F.3] and Young inequality, we obtain

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \|\frac{v_r}{r}\|^{\frac{3}{4}}_{L^6}\|\partial_z(\frac{v_r}{r})\|^{\frac{1}{4}}_{L^2}\|\sigma\|^{\frac{3}{2}}_{L^2}\|\nabla_h\sigma\|^{\frac{1}{2}}_{L^2}\\
&\lesssim \|\frac{\omega_\theta}{r}\|^{\frac{4}{3}}_{L^2}\|\sigma\|^2_{L^2} + \frac{1}{4}\|\nabla_h\sigma\|^2_{L^2} \qquad (5)
\end{aligned}
$$

Analogously, we find out

$$
\begin{aligned}
\mathrm{I}_3 &\lesssim \|\partial_r v_r\|^{\frac{3}{4}}_{L^6}\|\partial_z\partial_r v_r\|^{\frac{1}{4}}_{L^2}\|\sigma\|^{\frac{3}{2}}_{L^2}\|\nabla_h\sigma\|^{\frac{1}{2}}_{L^2}\\
&\lesssim \|\nabla_h\omega\|_{L^2}\|\sigma\|^2_{L^2} + \frac{1}{4}\|\nabla_h\sigma\|^2_{L^2} \qquad (6)
\end{aligned}
$$

For the last term I_4, we use Hölder's and Young's inequalities to get:

$$
\begin{aligned}
\mathrm{I}_4 &\lesssim \|\partial_z v_z\|_{L^2}\|\sigma\|_{L^2}\\
&\lesssim \|\nabla v\|^2_{L^2} + \|\sigma\|^2_{L^2}\\
&\lesssim \|\omega\|^2_{L^2} + \|\sigma\|^2_{L^2}. \qquad (7)
\end{aligned}
$$

Then by Integrating (4) with respect to time and using Grönwall's inequality to obtain the required result. In order to prove (3), we apply the partial derivative ∂_z to the vorticity equation, to obtain

$$
\partial_t\widetilde{\omega} + v\cdot\nabla\widetilde{\omega} - \Delta_h\widetilde{\omega} = \partial_z\partial_r\varrho e_\theta + \frac{\partial_z}{r}\omega - \partial_z v_r\partial_r\omega + \widetilde{\omega}\big(\frac{v_r}{r} - \partial_z v_z\big),
$$

with $\widetilde{\omega} = \partial_z\omega$. We have

$$
\partial_t\widetilde{\omega} + v\cdot\nabla\widetilde{\omega} - \Delta_h\widetilde{\omega} = -\partial_z\partial_r\varrho e_\theta + \frac{\partial_z v_r}{r}\omega - \partial_z v_r\partial_r\omega + \widetilde{\omega}\big(2\frac{v_r}{r} + \partial_r v_r\big).
$$

The classical L^2-estimate leads to

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\widetilde{\omega}(t)\|^2_{L^2} + \|\nabla_h\widetilde{\omega}(t)\|^2_{L^2} &= -\int_{\mathbb{R}^3}\partial_z\partial_r\varrho e_\theta\widetilde{\omega}dx + \int_{\mathbb{R}^3}\frac{\partial_z v_r}{r}\omega\widetilde{\omega}dx \qquad (8)\\
&- \int_{\mathbb{R}^3}\partial_z v_r\partial_r\omega\widetilde{\omega}dx + \int_{\mathbb{R}^3}2\frac{v_r}{r}\widetilde{\omega}dx + \int_{\mathbb{R}^3}\partial_r v_r\widetilde{\omega}dx\\
&= \sum_{i=1}^{5}\mathrm{J}_i.
\end{aligned}
$$

Combining Cauchy-Schwartz's inequality with Young's inequality we obtain

$$
\begin{aligned}
\mathrm{J}_1 &\leq \|\partial_z\partial_r\varrho\|_{L^2}\|\widetilde{\omega}\|_{L^2}\\
&\leq \|\partial_z\partial_r\varrho\|^2_{L^2} + \|\widetilde{\omega}\|^2_{L^2} \leq \|\nabla_h\partial_z\varrho\|^2_{L^2} + \|\widetilde{\omega}\|^2_{L^2}. \qquad (9)
\end{aligned}
$$

To control J_2 we make use the anisotropic inequality [5, Lemma F.5] to obtain

$$\begin{aligned} J_2 &\lesssim \|\partial_z \frac{v_r}{r}\|_{L^2}\|\omega\|_{L^6}^{\frac{3}{4}}\|\widetilde{\omega}\|_{L^2}^{\frac{1}{4}}\|\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\nabla_h\widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{3}{4}}\|\nabla_h\omega\|_{L^2}^2 + \|\frac{\omega_\theta}{r}\|_{L^2}^{\frac{3}{4}}\|\widetilde{\omega}\|_{L^2}^2 + \frac{1}{8}\|\nabla_h\widetilde{\omega}\|_{L^2}^2. \end{aligned}$$

From ([5], Lemma F.2), we find

$$\begin{aligned} J_3 &\lesssim 2\|\frac{v_r}{r}\|_{L^2}\|\partial_z\frac{v_r}{r}\|_{L^2}^{\frac{1}{4}}\|\widetilde{\omega}\|_{L^2}^{\frac{3}{2}}\|\nabla_h\widetilde{\omega}\|_{L^2} \\ &\lesssim \|\frac{\omega}{r}\|_{L^2}^{\frac{3}{4}}\|\widetilde{\omega}\|_{L^2}^2 + \frac{1}{8}\|\nabla_h\widetilde{\omega}\|_{L^2}^2. \end{aligned} \tag{10}$$

Similarly, we obtain

$$\begin{aligned} J_4 &\lesssim \|\partial_r\omega\|_{L^2}^{\frac{1}{2}}\|\partial_r\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_z v_r\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_z v_r\|_{L^2}^{\frac{1}{2}}\|\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\nabla_h\widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\nabla_h\omega\|_{L^2}^2\|\omega\|_{L^2}^2\|\widetilde{\omega}\|_{L^2}^2 + \|\nabla_h\partial_z v_r\|_{L^2}^2 + \frac{1}{8}\|\nabla_h\widetilde{\omega}\|_{L^2}^2 \end{aligned}$$

For the last term, we get

$$\begin{aligned} J_5 &\lesssim \|\partial_r v_r\|_{L^6}^{\frac{3}{4}}\|\partial_z\partial_r v_r\|_{L^2}^{\frac{1}{4}}\|\widetilde{\omega}\|_{L^2}^{\frac{3}{2}}\|\nabla_h\widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\nabla_h\omega\|_{L^2}\|\widetilde{\omega}\|_{L^2}^2 + \frac{1}{8}\|\nabla_h\widetilde{\omega}\|_{L^2}^2. \end{aligned} \tag{11}$$

Putting the above inequalities together and using Grönwall's inequality we get the desired result. □

Proposition 3. *Assume that $v_0 \in L^2$,with $\frac{\omega^\theta}{r} \in L^2$ and $\varrho_0 \in L^2$ .Let be a smooth axisymmetric solution of (B_{v_h,κ_h}) without swirl , then we have*

$$\|\frac{\omega_\theta}{r}(t)\|_{L^2}^2 + \int_0^t \|\nabla_h\frac{\omega_\theta}{r}(\tau)\|_{L^2}^2 d\tau \le 2(\|v_0\|_{L^2} + \|\varrho_0\|_{L^2})^2\Big(\|\frac{\omega_\theta}{r}(0)\|_{L^2} + \|\varrho_0\|_{L^2} + 2\Big)^2 e^{Ct}.$$

Proof. According to (3) the coupled function $\widetilde{\Gamma} \triangleq \frac{\omega^\theta}{r} - \frac{\varrho}{2}$ satisfies the following equation

$$\Big(\partial_t + v\cdot\nabla - (\Delta_h + \frac{2}{r}\partial_r)\Big)\widetilde{\Gamma} = -\frac{v^z}{2}.$$

On one hand, from the classical L^2-estimate, we obtain

$$\|\widetilde{\Gamma}(t)\|_{L^2}^2 + \|\nabla_h\widetilde{\Gamma}\|_{L_t^2L^2}^2 \le (\|v_0\|_{L^2} + \|\varrho_0\|_{L^2})^2\Big(\|\widetilde{\Gamma}_0\|_{L^2} + 1\Big)^2 e^{Ct}. \tag{12}$$

On the other hand, we have

$$\begin{aligned} \|\frac{\omega^\theta}{r}(t)\|_{L^2}^2 + \int_0^t \|\nabla_h\frac{\omega^\theta}{r}(\tau)\|_{L^2}^2 d\tau &\le (\|\widetilde{\Gamma}(t)\|_{L^2} + \|\varrho(t)\|_{L^2})^2(\|\nabla_h\widetilde{\Gamma}(t)\|_{L^2} + \|\nabla_h\varrho(t)\|_{L_t^2L^2})^2 \\ &\le 2(\|v_0\|_{L^2} + \|\varrho_0\|_{L^2})^2\Big(\|\widetilde{\Gamma}_0\|_{L^2} + 2\Big)^2 e^{Ct}. \end{aligned}$$

This gives the frist estimate. For the second one, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h\omega_\theta(t)\|_{L^2}^2 + \|\frac{\omega_\theta}{r}(t)\|_{L^2}^2 = \int_{\mathbb{R}^3}\frac{v_r}{r}\omega_\theta\omega_\theta dx - \int_{\mathbb{R}^3}\partial_r\varrho\omega_\theta.$$

By an integration by parts, we find out

$$\begin{aligned}\int_{\mathbb{R}^3} \partial_r \varrho \omega_\theta dx &= 2\pi \int \varrho \omega_\theta r dr dz = 2\pi \int \varrho \partial_r \omega_\theta r dr dz + 2\pi \int \varrho \omega_\theta dr dz \\ &= \int_{\mathbb{R}^3} \varrho \partial_r \omega_\theta dx + \int_{\mathbb{R}^3} \varrho \frac{\omega_\theta}{r} dx.\end{aligned}$$

By combining with Hölder and Young's inequality, we infer

$$\begin{aligned}\Big| \int_{\mathbb{R}^3} \partial_r \varrho \omega_\theta dx \Big| &\leq \|\varrho(t)\|_{L^2}(\|\nabla_h \omega_\theta(t)\|_{L^2} + \|\omega_\theta / r(t)\|_{L^2}) \\ &\leq 2(\|v_0\|_{L^2} + \|\varrho_0\|_{L^2})^2 e^{Ct} + \frac{1}{4}(\|\nabla_h \omega_\theta(t)\|^2_{L^2} + \|\omega_\theta / r(t)\|^2_{L^2}).\end{aligned}$$

On other hand, we get

$$\begin{aligned}\Big| \int_{\mathbb{R}^3} \varrho \frac{\omega_\theta}{r} dx \Big| &\leq \|v^r\|_{L^\infty} \|\frac{\omega_\theta}{r}\|_{L^2} \|\omega_\theta\|_{L^2} \\ &\leq C \|\frac{\omega_\theta}{r}\|^{\frac{4}{3}}_{L^2} \|\omega_\theta\|^2_{L^2} + \frac{1}{4} \|\nabla_h \omega_\theta\|^2_{L^2}.\end{aligned}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|\omega_\theta(t)\|^2_{L^2} + \|\nabla_h \omega_\theta(t)\|^2_{L^2} + \|\frac{\omega_\theta}{r}(t)\|^2_{L^2} \lesssim (\|v_0\|_{L^2} + \|\varrho_0\|_{L^2})^2 e^{Ct} + \|\frac{\omega_\theta}{r}\|^{\frac{4}{3}}_{L^2} \|\omega_\theta\|^2_{L^2}.$$

Integrating the above inequality with respect to time, we get

$$\begin{aligned}\|\omega_\theta(t)\|^2_{L^2} + \int_0^t \Big(\|\nabla_h \omega_\theta(\tau)\|^2_{L^2} + \|\frac{\omega_\theta}{r}(\tau)\|^2_{L^2} \Big) d\tau &\lesssim \|\omega_\theta(0)\|^2_{L^2} + (\|v_0\|_{L^2} + \|\varrho_0\|_{L^2})^2 e^{Ct} \\ &+ \int_0^t \|\frac{\omega_\theta}{r}(\tau)\|^{\frac{4}{3}}_{L^2} \|\omega_\theta(\tau)\|^2_{L^2} d\tau.\end{aligned}$$

Since $\|\omega\|_{L^2} = \|\omega_\theta\|_{L^2}$ and $\|\nabla_h \omega(t)\|^2_{L^2} = \|\nabla_h \omega_\theta\|^2_{L^2} + \|\frac{\omega_\theta}{r}\|^2_{L^2}$, then

$$\|\omega(t)\|^2_{L^2} + \int_0^t \|\nabla_h \omega(\tau)\|^2_{L^2} \leq C_0 e^{C_0 t}. \tag{13}$$

□

Proposition 4. *Let (v, ϱ) be a smooth solution of (B_{v_h,κ_h}) such that $v_0 \in H^1$ be divergence free axisymmetric without swirl vector field ,with $\frac{\omega^\theta}{r} \in L^2, \partial_z \omega^0 \in L^2$ and $\varrho_0 \in H^{0,1}$ be an axisymmetric function.Then we have*

$$\|\nabla v\|_{L^1_t L^\infty} \leq C_0 e^{exp(C_0 t)}.$$

Proof. By using the fact div$v = \partial_r v^r + \frac{v^r}{r} + \partial_z v^z = 0$ and $\omega^\theta = \partial_z v^r - \partial_r v^z$, we obtain

$$\begin{aligned}\|\nabla v\|_{L^1_t L^\infty} &\leq \|\partial_r v^r\|_{L^1_t L^\infty} + \|\frac{v^r}{r}\|_{L^1_t L^\infty} + \|\partial_z v^z\|_{L^1_t L^\infty} \\ &\lesssim \|\frac{v^r}{r}\|_{L^1_t L^\infty} + \|\partial_r v^r\|_{L^1_t L^\infty} + \|\partial_z v^r\|_{L^1_t L^\infty} + \|\partial_r v^z\|_{L^1_t L^\infty}.\end{aligned}$$

We have:

$$\|\frac{v^r}{r}\|_{L^1_t L^\infty} \lesssim \|\frac{\omega_\theta}{r}\|^{\frac{1}{2}}_{L^1_t L^\infty} \|\nabla_h \Big(\frac{\omega_\theta}{r}\Big)\|^{\frac{1}{2}}_{L^1_t L^\infty} \lesssim C_0 e^{(C_0 t)}.$$

Also

$$\|\partial_z v^r\|_{L^1_t L^\infty} \lesssim \int_0^t \|\nabla \partial_z v^r\|^{\frac{1}{2}}_{L^2} \|\nabla_h \partial_z v^r\|^{\frac{1}{2}}_{L^2} d\tau \lesssim t^{\frac{1}{2}} \|\partial_z \omega\|^{\frac{1}{2}}_{L^\infty_t L^2} \|\nabla_h \partial_z \omega\|^{\frac{1}{2}}_{L^1_t L^2}.$$

For the last quantities, we have

$$\|\partial_r v^z\|_{L^1_t L^\infty} + \|\partial_r v^r\|_{L^1_t L^\infty} \lesssim \|v\|_{L^1_t B^{1,\frac{1}{2}}_{2,1}}.$$

Therefore,

$$\begin{aligned}\|v\|_{L^1_t B^{1,\frac{1}{2}}_{2,1}} &\lesssim \|v_0\|_{B^{0,\frac{1}{2}}_{2,1}} + \|v\|_{L^1_t B^{0,\frac{1}{2}}_{2,1}} + \|v\otimes v\|_{L^1_t B^{1,\frac{1}{2}}_{2,1}} + \|v\otimes v\|_{L^1_t B^{0,\frac{3}{2}}_{2,1}} + \|\varrho\|_{L^1_t B^{0,\frac{1}{2}}_{2,1}} \\ &\lesssim \|v_0\|_{H^{1,1}} + \|v\|_{L^1_t H^{1,1}} + \|v\otimes v\|_{L^1_t H^{2,1}} + \|v\otimes v\|_{L^1_t H^{\frac{5}{4},\frac{7}{4}}} + \|\varrho\|_{L^1_t H^{1,1}} \\ &\lesssim \|v_0\|_{H^{1,1}} + \|v\|_{L^2_t H^{1,1}} + \|v\|^2_{L^1_t H^{2,1}} + \|v\|^2_{L^2_t H^{\frac{5}{4},\frac{7}{4}}} + \|\varrho\|_{L^1_t H^{1,1}},\end{aligned}$$

for the definition of Besov spaces and Anisotropic Besov spaces, we refer the reader to ([5], Definition.2.2). Hence

$$\|v\|_{L^2_t H^{2,1}} \lesssim \|v\|_{L^2_t L^2} + \|\partial_z v\|_{L^2_t L^2} + \|\nabla_h^2 v\|_{L^2_t L^2} + \|\nabla_h^2 \partial_z v\|_{L^2_t L^2}$$

and

$$\begin{aligned}\|v\|_{L^2_t H^{\frac{5}{4},\frac{7}{4}}} &\lesssim \|v\|_{L^2_t L^2} + \|\Lambda_h^{\frac{5}{4}} v\|_{L^2_t L^2} + \|\Lambda_\vartheta^{\frac{7}{4}} v\|_{L^2_t L^2} + \|\Lambda_h^{\frac{5}{4}}\Lambda_\vartheta^{\frac{7}{4}} v\|_{L^2_t L^2} \\ &\lesssim \|v\|_{L^2_t L^2} + \|\nabla_h v\|_{L^2_t L^2} + \|\nabla_h^2 v\|_{L^2_t L^2} + \|\partial_z v\|_{L^2_t L^2} + \|\partial_z^2 v\|_{L^2_t L^2} + \|\nabla_h \partial_z \omega\|_{L^2_t L^2}.\end{aligned}$$

For the term $\|\varrho\|_{L^1_t H^{1,1}}$, we have

$$\|\varrho\|_{L^1_t H^{1,1}} \lesssim \|\varrho\|_{L^1_t L^2} + \|\partial_z \varrho\|_{L^1_t L^2} + \|\nabla_h \varrho\|_{L^1_t L^2} + \|\nabla_h \partial_z \varrho\|_{L^1_t L^2} \le C_0 e^{exp(C_0 t)},$$

then we get the desired result. □

4. Conclusions

In this paper, we were able to obtain the global well posedness for the anisotropic Boussinesq system with horizontal dissipation.

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