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Growth of Solutions of The Homogeneous Differential-Difference Equations †

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Abstract: In this article, we study the growth properties of solutions of homogeneous linear differential-difference equations in the whole complex plane $\sum_{j=0}^{n} A_j(z) f^{(j)}(z+c_j) = 0, n \in \mathbb{N}^+$, where $c_j, j = 0, \ldots, n$ are complex numbers, and $A_j(z), j = 0, \ldots, n$ are entire functions of the same order.

Keywords: meromorphic function; differential equation; difference equation; differential-difference equation

1. Introduction

Throughout this paper, we use the standard notations of the value distribution theory of meromorphic functions founded by Nevanlinna, see ([4,6,8]). We denote respectively by $\rho(f)$ and $\lambda(f)$ the order of growth and the exponent of convergence of the zeros of a meromorphic function f.

In [7], Lan and Chen have studied the growth and oscillation of meromorphic solutions of homogeneous complex linear difference equation

$$\sum_{j=1}^{n} A_j(z) f(z+c_j) = 0, n \in \mathbb{N}^+,$$

where $n \in \mathbb{N}^+$, $A_j(z)$ $(j = 1, \ldots, n)$ are entire functions and $c_j, j = 1, \ldots, n$ are distinct complex numbers. Under some conditions on the coefficients, they obtained estimation of the order of growth of meromorphic solutions and studied the relationship between the exponent of convergence of zeros and the order of growth of the entire solutions of the above linear difference equation.

2. Main Results

In this paper, we improve and extend the main results of Lan and Chen. Especially, we study the growth of meromorphic solutions of equation

$$\sum_{j=0}^{n} A_j(z) f^{(j)}(z+c_j) = 0, n \in \mathbb{N}^+. \quad (1)$$

The key result here is the difference analogue of the lemma on the logarithmic derivative obtained independently by Hulburd-Korhonen [5] and Chiang-Feng [2]. In fact, we prove the following two results.

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Theorem 1. *Let* $c_j, j = 0, \dots, n$ *be complex constants and let*

$$A_j(z) = P_j(z)e^{h_j(z)} + Q_j(z), \tag{2}$$

where $h_j(z) = a_{jk}z^k + a_{jk-1}z^{k-1} + \cdots + a_{j0}$ *are polynomials of degree* $k \geq 1$ *and* $P_j(z)(\not\equiv 0)$ *and* $Q_j(z)$ *are entire functions whose order is lower than* k. *Suppose that* $|a_{0k}| > \max\limits_{1\leq j\leq n}\{|a_{jk}|\}$. *If* $f(z)(\not\equiv 0)$ *is a meromorphic solution of equation (1), then* $f(z) \geq k+1$.

Example 1. *The function* $f(z) = e^{z^2}$ *is a solution of equation*

$$A_2(z)f''(z+i) + A_1(z)f'(z+i) + A_0(z)f(z-4i) = 0,$$

where

$$A_0(z) = -(4iz^3 - 4i\pi z^2 + 2z(4\pi + i) + 2i\pi)e^{8iz+16},$$

$$A_1(z) = -[(2z^3 - z)e^{-2iz+1} - (2z^2 + 4iz - 1)],$$

$$A_2(z) = (z^2 - i\pi)e^{-2iz+1} - (z+i),$$

$$P_0(z) = -(4iz^3 - 4i\pi z^2 + 2z(4\pi + i) + 2i\pi), h_0(z) = 8iz + 16, Q_0(z) \equiv 0,$$

$$P_1(z) = -(2z^3 - z), h_1(z) = -2iz + 1, Q_1(z) = -(2z^2 + 4iz - 1),$$

$$P_2(z) = z^2 - i\pi, h_2(z) = -2iz + 1, Q_2(z) = -(z+i).$$

Furthermore, $\rho(P_j(z)) = 0$, $j = 0, 1, 2$ *and*

$$|a_{01}| = |8i| = 8 > max\{|a_{11}|, |a_{21}|\} = max\{|-2i|, |-2i|\} = 2.$$

Hence, the conditions of Theorem 1 are satisfied. We see that for $j = 0, 1, 2$

$$\rho(f) = 2 = \rho(A_j) + 1 = \deg(h_j) + 1 = 1 + 1 = 2.$$

Corollary 1. *Let* $k, A_j(z)(\not\equiv 0), j = 0, \dots, n$ *satisfy the assumptions of Theorem 1, let* $B_i(z)$, $i = 1, \dots, m$ *be entire functions whose order is lower than* k, *and let* c_j, $j = 0, \dots, n+m$ *be complex constants. If* $f(z)(\not\equiv 0)$ *is a meromorphic solution of equation*

$$\begin{gathered} B_m(z)f^{(n+m)}(z + c_{n+m}) + \cdots + B_1(z)f^{(n+1)}(z + c_{n+1}) + A_n(z)f^{(n)}(z + c_n) + \cdots + \\ A_1(z)f'(z + c_1) + A_0(z)f(z + c_0) = 0, \end{gathered} \tag{3}$$

then $\rho(f) \geq k+1$.

Example 2. *The function* $f(z) = e^{2z^2}$ *is a solution of equation*

$$B_1(z)f''(z+2) + A_1(z)f'(z-1) + A_0(z)f(z+2) = 0,$$

where

$$B_1(z) = z^4 - i\pi,$$

$$A_1(z) = -(4z^5 + i\pi z^2)e^{4z-2},$$

$$A_0(z) = 4z^2(4z^4 - 4z^3 + i\pi(z-1))e^{-8z-8} - 4[4z^6 + 16z^5 + 17z^4 - i\pi(4z^2 + 16z + 17)],$$

$$P_0(z) = 4z^2(4z^4 - 4z^3 + i\pi(z-1)), h_0(z) = -8(z-1),$$

$$Q_0(z) = -4[4z^6 + 16z^5 + 17z^4 - i\pi(4z^2 + 16z + 17)],$$

$$P_1(z) = -(4z^5 + i\pi z^2), h_1(z) = 4z - 2, Q_1(z) \equiv 0.$$

Moreover, $\rho(P_0) = \rho(P_1) = 0$ *and*

$$\rho(B_1) = 0 < \rho(A_1) = \rho(A_2) = 1,$$

$$|a_{01}| = |-8| = 8 > |a_{11}| = |4| = 4.$$

Thus, the conditions of Corollary 1 are satisfied. We see that for $j = 0, 1$

$$\rho(f) = 2 = \rho(\mathrm{A_j}) + 1 = \deg(\mathrm{h_j}) + 1 = 2.$$

3. Preliminary Lemmas

For the proof of our results, we need the following lemmas.

Lemma 1. *([1]) Suppose that* $\boldsymbol{f(z)}$ *is a meromorphic function with* $\boldsymbol{\rho(f) = \rho} < +\infty$*. Then, for any given* $\boldsymbol{\varepsilon > 0}$*, one can find a set* $\boldsymbol{E} \in (\mathbf{1}, +\infty)$ *of finite linear measure or finite logarithmic measure such that*

$$|\boldsymbol{f(z)}| \le \boldsymbol{e}^{\boldsymbol{r}^{\boldsymbol{\rho+\varepsilon}}}$$

holds for all z satisfying $|z| = r \notin [0,1] \cup E$ *as* $r \to +\infty$*.*

Lemma 2. *([2]) Let* η_1*,* η_2 *be two arbitrary complex numbers and let* $f(z)$ *be a meromorphic function of finite order* ρ*. For any given* $\varepsilon > 0$*, there exists a subset* $E \in (0, +\infty)$ *of finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E$*, the following double inequality holds*

$$e^{-r^{\rho-1+\varepsilon}} \le \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \le e^{r^{\rho-1+\varepsilon}}.$$

Lemma 3. *([3]) Let* $f(z)$ *be a transcendaental meromorphic function of finite order* ρ*, and let* $\varepsilon > 0$ *be a given constant. Then, there exists a subset* $E \in (1, +\infty)$ *that has finite logarithmic measure, such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E$*, and for all* $k, j, 0 \le j < k$*, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le r^{(k-j)(\rho-1+\varepsilon)}.$$

4. Proofs

Proof of Theorem 1. Contrary to our assertion, we assume that $\rho = \rho(f) < k + 1$. Let

$$h_j(z) = a_{jk}z^k + h_j^*(z), \tag{4}$$

where $a_{jk} \neq 0$ are complex constants and $h_j^*(z)$ are polynomials with $\deg h_j^* \le k - 1, j = 0, \dots, n$. We set

$$|a_{0k}| > |a_{jk}|, \qquad \theta_0 \neq \theta_j, \qquad \theta_j = \arg(a_{jk}) \in [0, 2\pi), \qquad 1 \le j \le n.$$

We now choose θ such that

$$\cos(k\theta + \theta_0) = 1. \tag{5}$$

Thus, by $\theta_j \neq \theta_0$, $1 \le j \le n$, we find

$$\cos(k\theta + \theta_j) < 1, \ \ 1 \le j \le n. \tag{6}$$

Denote

$$a = |a_{0k}|, \qquad b = \max_{1\le j\le n}\{|a_{jk}|\}, \qquad c = \max_{1\le \mathrm{j} \le \mathrm{n}}\{b\cos(k\theta + \theta_j)\} < a \tag{7}$$

and

$$\beta = \max_{0\le j\le n}\{\rho(P_j),\rho(Q_j)\} < k. \tag{8}$$

Clearly

$$\rho\left(\frac{P_j}{P_0}\right) \le \max_{1\le j\le n}\{\rho(P_j),\rho(P_0)\} \le \beta, \quad \rho\left(\frac{Q_j}{P_0}\right) \le \max_{0\le j\le n}\{\rho(P_0),\rho(Q_j)\} \le \beta.$$

By Lemma 1, for any given ε satisfying

$$0 < 2\varepsilon < \min\{1, k+1-\rho, k-\beta, a-c\},$$

there is a set $E_1 \subset (1,+\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{P_j(z)}{P_0(z)}\right| \le e^{r^{\beta+\varepsilon}}, \qquad 1 \le j \le n, \qquad \left|\frac{Q_j(z)}{P_0(z)}\right| \le e^{r^{\beta+\varepsilon}}, \qquad 0 \le j \le n. \tag{9}$$

By the definition of the order of entire function, for any given $\varepsilon > 0$ and all sufficiently large $z, |z| = r$, we get

$$\left|e^{-h_0^*(z)}\right| \le e^{r^{k-1+\varepsilon}}, \qquad \left|e^{h_j^*(z)}\right| \le e^{r^{k-1+\varepsilon}}, \qquad 1 \le j \le n. \tag{10}$$

Applying Lemmas 2 and 3 to $f(z)$, we conclude that there is a set $E_2 \subset (1,+\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, we have for $1 \le j \le n$

$$\left|\frac{f^{(j)}(z+c_j)}{f(z+c_0)}\right| = \left|\frac{f^{(j)}(z+c_j)}{f(z+c_j)}\frac{f(z+c_j)}{f(z+c_0)}\right| \le r^{j(\rho-1+\varepsilon)}e^{r^{\rho-1+\varepsilon}}. \tag{11}$$

By substituting (2) into Equation (1), we obtain

$$\left|-e^{a_{0k}z^k}\right| \le \left|\sum_{j=1}^{n} e^{-h_0^*(z)}\frac{f^{(j)}(z+c_j)}{f(z+c_0)}\left(\frac{P_j(z)}{P_0(z)}e^{a_{jk}z^k+h_j^*(z)} + \frac{Q_j(z)}{P_0(z)}\right)\right| + \left|e^{-h_0^*(z)}\frac{Q_0(z)}{P_0(z)}\right|. \tag{12}$$

Let $z = re^{i\theta}$, where $r \notin [0,1] \cup E_1 \cup E_2$. Substituting (5)–(7) and (9)–(11) into (12), we find

$$e^{ar^k} \le \sum_{j=1}^{n} r^{j(\rho-1+\varepsilon)}e^{r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\beta+\varepsilon}}\left(e^{b\cos(k\theta+\theta_j)r^k+r^{k-1+\varepsilon}} + 1\right) + e^{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}},$$

Thus for $0 < 2\varepsilon < \min\{1, k+1-\rho, k-\beta, a-c\}$, we obtain

$$e^{ar^k} \le (n+1)r^{n(\rho-1+\varepsilon)}e^{(c+\varepsilon)r^k+2r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\beta+\varepsilon}} \le (n+1)r^{n(\rho-1+\varepsilon)}e^{(c+2\varepsilon)r^k}. \tag{13}$$

Dividing both sides of (13) by $(n+1)r^{n(\rho-1+\varepsilon)}e^{(c+2\varepsilon)r^k}$ and letting $r \to +\infty$, since $0 < 2\varepsilon < a-c$, we get $+\infty \le 1$. This is a contradiction, hence $\rho(f) \ge k+1$. □

Proof of Corollary 1

Assume that $\rho = \rho(f) < k+1$. By using the similar steps as in the proof of Theorem 1, we also obtain (4)–(10). By Lemma 1, there is a set $E_3 \subset (1,+\infty)$ with finite logarithmic measure such that, for any given $\varepsilon > 0$ and all z satisfying $|z| = r \notin [0,1] \cup E_3$, we get

$$\left|B_j(z)\right| \le e^{r^{\beta_1+\varepsilon}}, \qquad 1 \le j \le m,$$

where

$$\beta_1 = \max_{1\le j\le m}\{\rho(B_j)\} < k.$$

We take

$$\gamma = \max_{1\le j\le m}\{\rho(B_j(z),\rho(P_0)\} < k, \qquad \rho\left(\frac{B_j(z)}{P_0(z)}\right) \le \max_{1\le j\le m}\{\rho(B_j),\rho(P_0)\}.$$

And by applying Lemmas 2 and 3 to $f(z)$ we conclude that there is a set $E_4 \subset (1,+\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have for $1 \leq j \leq n+m$

$$\left|\frac{f^{(j)}(z+c_j)}{f(z+c_0)}\right| = \left|\frac{f^{(j)}(z+c_j)}{f(z+c_j)}\frac{f(z+c_j)}{f(z+c_0)}\right| \leq r^{j(\rho-1+\varepsilon)}e^{r^{\rho-1+\varepsilon}}, \tag{14}$$

and

$$\left|\frac{B_j(z)}{P_0(z)}\right| \leq e^{r^{\gamma+\varepsilon}}, \qquad n+1 \leq j \leq n+m. \tag{15}$$

By substituting (2) into (3), we find

$$\begin{aligned}\left|-e^{a_{0k}z^k}\right| \leq & \left|\sum_{j=1}^{n} e^{-h_0^*(z)}\frac{f^{(j)}(z+c_j)}{f(z+c_0)}\left(\frac{P_j(z)}{P_0(z)}e^{a_{jk}z^k+h_j^*(z)} + \frac{Q_j(z)}{P_0(z)}\right)\right| + \\ & \left|\sum_{j=n+1}^{m+n} e^{-h_0^*(z)}\frac{f^{(j)}(z+c_j)}{f(z+c_0)}\frac{B_{j-n}(z)}{P_0(z)}\right| + \left|e^{-h_0^*(z)}\frac{Q_0(z)}{P_0(z)}\right|.\end{aligned} \tag{16}$$

Let $z = re^{i\theta}$, where $r \notin [0,1] \cup E_1 \cup E_2 \cup E_3 \cup E_4$. Substitutying (5)–(7), (9)–(10), (14) and (15) into (16) we obtain

$$\begin{aligned} e^{ar^k} \leq & \left|\sum_{j=1}^{n} r^{j(\rho-1+\varepsilon)}e^{r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\beta+\varepsilon}}\left(e^{b\cos(k\theta+\theta_j)r^k+r^{k-1+\varepsilon}}+1\right)\right| \\ & + \sum_{j=n+1}^{m+n} r^{j(\rho-1+\varepsilon)}e^{r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\gamma+\varepsilon}} + e^{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}},\end{aligned}$$

thus

$$\begin{aligned} e^{ar^k} & \leq nr^{n(\rho-1+\varepsilon)}e^{(c+\varepsilon)r^k+2r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\beta+\varepsilon}} \\ & +mr^{(m+n)(\rho-1+\varepsilon)}e^{r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\gamma+\varepsilon}} + e^{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}} \leq nr^{n(\rho-1+\varepsilon)}e^{(c+2\varepsilon)r^k} \\ & +mr^{(m+n)(\rho-1+\varepsilon)}e^{r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\gamma+\varepsilon}} + e^{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}}.\end{aligned} \tag{17}$$

Dividing both sides of (17) by e^{ar^k} and letting $r \to +\infty$, we obtain $1 \leq 0$ since $0 < 2\varepsilon < \min\{1, k+1-\rho, k-\beta, a-c, k-\gamma\}$. This is a contradiction, then $\rho(f) \geq k+1$. □

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