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Proceeding Paper

Existence and Attractivity Results for Fractional Differential Inclusions via Nocompactness Measures †

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Abstract: In this paper, we use the concept of measure of nocompactness and fixed point theorems to investigate the existence and stability of solutions of a class of Hadamard-Stieltjes fractional differential inclusion in an appropriate Banach space, these results are proven under sufficient hypotheses. We also give an example to illustrate the obtained results.

Keywords: fractional differential inclusions; nocompactness measures; regulated functions; stability; fixed-point theorems

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1. Introduction

In this work, we consider the following differential inclusion with initial condition

$$\begin{cases} \mathcal{D}^{\gamma}\left(\frac{dw}{d\phi}\right)(z) \in G(z, w(z)); \quad z \in (1, +\infty) \\ \frac{dw}{d\phi}(1) = w_0; \quad w(1) = w_1 \end{cases} \tag{1}$$

where $w_0, w_1 \in \mathbb{R}$, $\frac{dw}{d\phi}$ is the Stieltjes derivative of w with respect to ϕ, $\mathcal{D}^{\gamma}$ is the Hadamard fractional derivative of order $0 < \gamma < 1$, $G : [1, +\infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is multivalued map and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

2. Preliminaries

Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is monotone, nondecreasing and continuous from the left everywhere. D_ϕ is the set of discontinuity points of ϕ. We denote by $\mathcal{B}(J)$, the Banach space of bounded functions on the interval $J = [1, +\infty)$ equipped with the norm of uniform convergence, and by $\mathcal{B}_\phi(J)$ the subspace of bounded functions which are also ϕ-continuous on J.

Theorem 1. *$\mathcal{B}_\phi(J)$ is a Banach space.*

Let X and Y be two Banach spaces, we define

$$\mathcal{P}_{cl,cv,cp}(X) = \{\Omega \in \mathcal{P}(X);\ \Omega \text{ is closed,convex,compact}\}.$$

Theorem 2 ([10])**.** *Let $G : X \to \mathcal{P}_{cl,cv}(Y)$ be a lower semicontinuous multifunction. Then G admits continuous selection.*

Theorem 3 ([10])**.** *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space X with ψ is a measure of nocompactness in it and let $G : \Omega \longrightarrow \mathcal{P}_{cl,cv}(\Omega)$ be a closed. Assume that there exists a constant $k \in [0, 1)$ such that $\psi(GA) \le k\psi(A)$ for any nonempty subset A of Ω. Then G has a fixed point in the set Ω.*

Remark 1. *Let us denote by Fix G the set of all fixed points of the operator G which belong to* Ω. *The set Fix G belongs to the family ker* ψ *see [10].*

3. Main Results

Consider the function ψ defined on the family $\mathcal{M}_{\mathcal{B}_\phi(J)}$ by the formula

$$\psi(\Omega) = \max\Big\{\omega_\phi(\Omega), \omega_\phi^+(\Omega)\Big\} + \limsup_{z\to\infty} diam\, \Omega(z). \tag{2}$$

Theorem 4. *The mappings* ψ *is a measure of nocompactness in the space* $\mathcal{B}_\phi(J)$.

Consider the following inclusion

$$u(z) \in (Fu)(z);\ z \in J. \tag{3}$$

Definition 1. *The solution* $u = u(z)$ *of (3) is said to be globally attractive if for each solution* $v = v(z)$ *of (3) we have that*

$$\lim_{z\to\infty} \big(u(z) - v(z)\big) = 0.$$

In the case when this limit is uniform i.e when for each $\epsilon > 0$ *there exists* $T > 1$ *such that*

$$\big|u(z) - v(z)\big| < \epsilon, \tag{4}$$

for $z \geq T$, *we will say that solutions of (3) are uniformly globally attractive.*

Remark 2. *The kernel ker* ψ *consists of nonempty and bounded sets* Ω *such that functions from* Ω *are locally equiregulated on J and for each* $u, v \in \Omega$, *(4) hold.*

Let $\frac{dw}{d\phi}(z) = v(z)$ then $w(z) = w_0 + \int_1^z v(t)d\phi(t)$ and (1) can be written as

$$v(z) \in w_1 + \frac{1}{\Gamma(r)} \int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} G\left(t, w_0 + \int_1^t v(\tau)d\phi(\tau)\right)\frac{dt}{t}, \tag{5}$$

Denote $|G(z,u)| = \{|u|;\ u \in F(z,u)\}$ and $\|G(z,u)\| = H(G(z,u),0) = \sup\{|u|;\ u \in G(z,u)\}$. The differential inclusion (1) will be considered under the following assumptions:

(H_1) There exist continuous and bounded functions $p, q : J \to \mathbb{R}_+$ such that

$$H(G(z,u), G(z,v)) \leq p(z)|u - v|;\ z \in J,$$

for all $u, v \in \mathbb{R}$, and

$$\|G(z,0)\| = H(G(z,0),0) \leq q(z);\ z \in J.$$

(H_2) For every (z,w) in $J \times \mathbb{R}$, $G(z,w)$ is a nonempty convex and closed subset of $\mathbb{R}$.

(H_3) Assume that

$$p^* = \sup_{z\in J}\left|\frac{w_0}{\Gamma(\gamma)}\int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} p(t)dt\right| < \infty,$$

$$q^* = \sup_{z\in J}\left|\frac{1}{\Gamma(\gamma)}\int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} q(t)dt\right| < \infty,$$

$$p_\phi = \sup_{z\in J}\left|\frac{1}{\Gamma(\gamma)}\int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} p(t)[\phi(t) - \phi(1)]dt\right| < 1,$$

Remark 3. *As observed there, G is lower semicontinuous, hence G admits continuous selection (Theorem 2).*

Theorem 5. *Under assumptions* $(H_1) - (H_3)$ *The differential inclusion (1) has at least one solution* $u = u(z)$ *in the space* $\mathcal{B}_\phi(J)$. *Moreover, solutions of the differential inclusion (1) are globally attractive.*

Proof. Consider the multi-valued operator N defined on the space $\mathcal{B}_\phi(J)$ in the following way:

$$N : \mathcal{B}_\phi(J) \to \mathcal{P}(\mathcal{B}_\phi(J));$$

such that, for each $u \in \mathcal{B}_\phi(J)$

$$(Nw)(z) = \left\{ u \in \mathcal{B}_\phi(J) \mid u(z) = w_1 + \frac{1}{\Gamma(r)} \int_1^z \left(\ln \frac{z}{t}\right)^{\gamma-1} g\left(t, w_0 + \int_1^t w(\tau) d\phi(\tau)\right) \frac{dt}{t} \right\},$$

where g is a selctor of G. For each w in $\mathcal{B}_\phi(J)$, and for each function u in Nw, we have u is ϕ-continuous on J. Next, let us take an arbitrary function $w \in \mathcal{B}_\phi(J)$, $u \in Nu$ and fixed $z \in J$, we get

$$|u(z)| \leq |w_1| + p^* + \|w\| p_\phi + q^*.$$

So, the operator N is well defined. We take

$$\zeta = \frac{|w_1| + \dfrac{p^*}{\Gamma(\gamma)} + \dfrac{q^*}{\Gamma(\gamma)}}{1 - p_\phi}.$$

We deduce that the operator $N : B_\zeta \longrightarrow \mathcal{P}(B_\zeta)$. Further, the set Nu is closed and convex in B_ζ. Now, we take a nonempty $\Omega \subset B_\zeta$, for $T > 1, z_1, z_2 \in [1, T]$ with $z_1 < z_2$ and $|\phi(z_2) - \phi(z_1)| \leq \epsilon$ for each $\epsilon > 0$. Fix arbitrarily $w \in \Omega$ and $u \in Nw$, we obtain

$$\omega_\phi(N\Omega) = 0. \tag{6}$$

For $z_0 \in J \cap D_\phi$, fix $z \in (z_0, z_0 + \epsilon)$, we have

$$\omega_\phi^+(N\Omega) = 0. \tag{7}$$

From (6) and (7), we infer that the set $N\Omega$ is equiregulated on J. Further, for $w, w' \in \Omega$, $u \in Nw$ and $u' \in Nw'$ and an arbitrary fixed $z \in [1, T]$, we get

$$\limsup_{z \to \infty} diam\, N\Omega(z) \leq p_\phi \limsup_{z \to \infty} diam\, \Omega(z). \tag{8}$$

Consequently, in view of (6)–(8), we deduce that

$$\psi(N\Omega) \leq p_\phi \psi(\Omega).$$

Then, by the Theorem 3 the operator N has at least one fixed point in B_ζ which is a solution of differential inclusion (1). Moreover, taking into account the fact that the set $Fix\, N \in \ker \psi$ (Remark 1) and the characterization of sets belonging to $\ker \psi$ (Remark 2), we conclude that all solutions of (1) are globally attractive in the sense of Definition 1. □

4. Example

We consider the following differential inclusion

$$\begin{cases} \mathcal{D}^\gamma \left(\frac{dw}{d\phi}\right)(z) \in \left[\frac{e^{-z}}{|u(z)|+z}, \frac{(z-1)e^{-z}}{|u(z)|+z}\right] \subset \mathbb{R};\ z > 1, \\ \frac{dw}{d\phi}(1) = w_0; \quad w(1) = w_1. \end{cases} \tag{9}$$

It is clear that (9) can be written as the differential inclusion (1), where $\phi(\tau) = \arctan([\tau])$ (the symbol $[\tau]$ indicates the integer value of τ). Let us show that conditions $(H_1) - (H_3)$ hold. For $z \in J$ and $u, v \in \mathbb{R}$, we have

$$H(F(z,u), F(z,v)) \leq (z-1)e^{-z}|u-v|.$$

So $p(z) = (z-1)e^{-z}$, $F(z,0) = \left[\inf\left\{\frac{e^{-z}}{z}, \frac{(z-1)}{z}e^{-z}\right\}, \max\left\{\frac{e^{-z}}{z}, \frac{(z-1)}{z}e^{-z}\right\}\right]$, hence

$$\|F(z,0)\| = \max\left\{\frac{e^{-z}}{z}, \frac{(z-1)}{z}e^{-z}\right\} = q(z).$$

Notice that the functions p, q are continuous and bounded on J. Next, for a fixed $z \in J$, we have $p^* < 1$. This estimate show that p^* and q^* are finite quantities. Consequently from Theorem 5 the differential inclusion (9) has at least solution in the space $\mathcal{B}_\phi(J)$ and solutions of (9) are globally attractive.

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