computer sciences and mathematics forum

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*Proceeding Paper*

# A Way to Construct Commutative Hyperstructures †

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† Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.

**Abstract:** This article aims to create commutative hyperstructures, starting with a non-commutative group. So, we consider the starting group to be the dihedral group $D_n$, with $n$ is a natural number, $n > 1$, and we determine the HX groups associated with the dihedral group. For a fixed number $n$, We notice $\mathcal{G}_n = \left\{\mathcal{G}_{p_2}^{p_1}\ HX - \text{groups, for any } p_1, p_2 \in \mathbb{N}^* \text{ such that } n = p_1 p_2\right\}$ be the set of all HX groups. This paper analyzes this new structure's properties for particular cases when the dihedral group $D_4$ is the support group.

**Keywords:** HX-groups; dihedral group; commutativity; hyperstructures

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**Citation:** Sonea, A. A Way to Construct Commutative Hyperstructures. *Comput. Sci. Math. Forum* **2023**, *1*, 0. https://doi.org/

Academic Editor:

Published: 28 April 2023



## 1. Introduction

The algebraic term hyperstructure denotes an appropriate generalization of structures of classical algebras, such as group, semigroup, and ring. In classical algebraic structures, the composition of two elements is an element, and within the algebraic hyperstructure, the hyper composition law of two elements represents a set. F. Marty noticed many aspects of a factor group, which was the starting point within the theory of hypergroups. He introduced the concept of hypergroup in 1934 on the occasion of the Congress of Mathematicians from the Scandinavian countries. Hypergroups are studied from the point of theory but also for their applications in pure and applied mathematics problems: Geometry, topology, cryptography, code theory, graphs, hypergraphs, automata theory, fuzzy degree, probability, etc. [1]. The Chinese mathematician Mi introduced the notion of HX-groups After that, Honghai and Li Honxing contribute to the theory of HX groups by Corsini [2–4]. Also, Cristea analyzed the connection between HX groups and hypergroups [5]. In the article [6], we study the form of HX groups with the dihedral group $D_n$ as a group, where $n$ is a natural number greater than 3. support. Moreover, we analyzed the HX groups' commutativity degree and the Chinese hypergroup's fuzzy grade associated with them. We noticed a connection between the commutativity degree of HX groups related to the dihedral group and the commutativity degree of the dihedral group. In the article [7] we define the new concept of Neutro HX-groups.

## 2. Main Results

In this section, we recall the notions of an HX-group, and we define the new set forms by HX-groups [8].

**Definition 1.** *Let $(G, .)$ be a group and $\mathcal{G} \subset \mathcal{P}^*(G) \backslash \{\emptyset\}$, where $\mathcal{P}^*(G)$ is the set of nonempty subsets of $G$. An HX-group is a nonempty subset $H$ of $\mathcal{P}^*(G)$ which is a group with respect to the operation "$*$" defined by:*

$$\forall\ A,\ B \in \mathcal{G},\ A * B = \{a.b \mid a \in A,\ b \in B\}. \tag{1}$$

*We say that $\mathcal{G}$ has group $G$ as support.*

**Definition 2.** *Let $(\mathcal{G}, *)$ be an HX-group with the support $(G, .)$ and E be the identity of group $\mathcal{G}$. A Chinese hypergrupoid is a hyperstructure $< G^*, \oplus >$, where*

$$G^* = \bigcup_{A\in\mathcal{G}} A \text{ and } \forall\ (x,y) \in G^* \times G^*,\ x \oplus y = \bigcup_{\substack{x\in A\\ y\in B\\ \{A,B\}\subset\mathcal{G}}} A * B.$$

*The Construction of $\mathcal{G}_n$*

In the following, we will consider the set of all HX groups having the dihedral group $(D_4, \cdot)$ as support. The dihedral group $D_n$, where $n \in \mathbb{N}$, $n \geq 2$, is the group generated by rotation $\rho$ and symmetry $\sigma$ and satisfies the following properties:

$$\begin{aligned} \rho^n &= \sigma^2 = e; \\ \rho^k \cdot \sigma &= \sigma \cdot \rho^{n-k},\ \forall k \leq n,\ k \in \mathbb{N}. \end{aligned} \tag{2}$$

It is denoted by $D_n =< \rho, \sigma >$. So, in the article [6], we determine the form of HX groups associated with the dihedral group $D_n$, and we analyzed them in particular cases for $D_4$, $D_5$ and $D_6$. For a fixed number $n$, We notice

$$\mathcal{G}_n = \left\{ \mathcal{G}_{p_2}^{p_1} |\ HX - \text{groups, for any } p_1,\ p_2 \in \mathbb{N}^* \text{ such that } n = p_1 p_2 \right\}$$

be the set of all HX groups. We define the following hyperoperation

$$'' \circ '' : \mathcal{G}_n \times \mathcal{G}_n \to P^*(\mathcal{G}_n)$$

thus

$$\begin{aligned} \mathcal{G}_{p_2}^{p_1} \circ \mathcal{G}_{p'_2}^{p'_1} &= \left\{ \bigcup_{0\leq s\leq 2p_2-1} \left( \bigcup_{0\leq t\leq 2p'_2-1} C_{s,t} \right);\ C_{s,t} = X_s^{p_1} * Y_t^{p'_1};\ X_s^{p_1} \subseteq \mathcal{G}_{p_2}^{p_1},\ Y_t^{p'_1} \subseteq \mathcal{G}_{p'_2}^{p'_1} \right\}, \\ n &= p_1 p_2 = p'_1 p'_2,\ p_1, p_2,\ p'_1,\ p'_2 \in \mathbb{N}^*. \end{aligned}$$

## 3. Results and Discussion

In the following we present a particular cases, for $n = 4$ and we analyze the hyperstructure $(\mathcal{G}_4, \circ)$.

**Proposition 1.** *The hyperstructure $(\mathcal{G}_4, \circ)$ is a commutative structure, where*

$$\mathcal{G}_4 = \left\{ \mathcal{G}_{p_2}^{p_1} |\ HX - \text{groups, for any } p_1,\ p_2 \in \mathbb{N}^* \text{ such that } 4 = p_1 p_2 \right\}.$$

**Proof.** So, $(\mathcal{G}_4, \circ)$ is a commutative hyperstructure if and only if $\mathcal{G}_{p_2}^{p_1} \circ \mathcal{G}_{p'_2}^{p'_1} = \mathcal{G}_{p'_2}^{p'_1} \circ \mathcal{G}_{p'_2}^{p_1}$, for any $p_1$, $p_2$, $p'_1$, $p'_2 \in \mathbb{N}^*$ such that $4 = p_1 p_2 = p'_1 p'_2$. According with the article [6], the form of the HX groups associated with the dihedral group $D_4$ is the following:

$$\begin{aligned} \mathcal{G}_2^2 &= \{\{e, \rho^2\}, \{\rho, \rho^3\}, \{\sigma, \rho^2\sigma\}, \{\rho\sigma, \rho^3\sigma\}\}; \\ \mathcal{G}_1^4 &= \{\{e, \rho, \rho^2, \rho^3\}, \{\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}\}; \\ \mathcal{G}_4^1 &= \{\{e\},\ \{\rho\},\ \{\rho^2\},\ \{\rho^3\},\ \{\sigma\},\ \{\rho\sigma\},\ \{\rho^2\sigma\},\ \{\rho^3\sigma\}\}. \end{aligned}$$

Therefore,

$$\mathcal{G}_2^2 \circ \mathcal{G}_1^4 = \{C_{0,0}, C_{1,0}, C_{2,0}, C_{3,0}, C_{0,1}, C_{1,1}, C_{2,1}, C_{3,1}\}$$

and the sets $X_s^{p_1}, Y_t^{p'_1}$are next:

$$\begin{aligned} X_0^2 &= \{e, \rho^2\},\ X_1^2 = \{\rho, \rho^3\},\ X_2^2 = \{\sigma, \rho^2\sigma\},\ X_3^2 = \{\rho\sigma, \rho^3\sigma\}; \\ Y_0^4 &= \{e, \rho, \rho^2, \rho^3\},\ Y_1^4 = \{\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}. \end{aligned}$$

In the calculation of the elements $C_{i,j}$, $i \in \{0,1,2,3\}$, $j \in \{0,1\}$, we use the rules gives by (2)

$$\begin{aligned} C_{0,0} &= X_0^2 * Y_0^4 = \{e, \rho^2\} * \{e, \rho, \rho^2, \rho^3\} = e \cdot e \cup e \cdot \rho \cup e \cdot \rho^2 \cup \\ & \cup e \cdot \rho^3 \cup \rho^2 \cdot e \cup \rho^2 \cdot \rho \cup \rho^2 \cdot \rho^2 \cup \rho^2 \cdot \rho^3 \\ &= \{e, \rho, \rho^2, \rho^3\} = Y_0^4; \\ C_{1,0} &= X_1^2 * Y_0^4 = \{\rho, \rho^3\} * \{e, \rho, \rho^2, \rho^3\} = \rho \cdot e \cup \rho \cdot \rho \cup \rho \cdot \rho^2 \cup \rho \cdot \rho^3 \cup \\ & \cup \rho^3 \cdot e \cup \rho^3 \cdot \rho \cup \rho^3 \cdot \rho^2 \cup \rho^3 \cdot \rho^3. \\ &= \{e, \rho, \rho^2, \rho^3\} = Y_0^4 \\ C_{2,0} &= X_2^2 * Y_0^4 = \{\sigma, \rho^2\sigma\} * \{e, \rho, \rho^2, \rho^3\} = \{\sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3, \rho^2\sigma, \rho^2\sigma\rho, \rho^2\sigma\rho^2, \rho^2\sigma\rho^3\} \\ &= \{\sigma, \rho^3\sigma, \rho^2\sigma, \rho\sigma\} = \{\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\} = Y_1^4. \\ C_{3,0} &= X_3^2 * Y_0^4 = \{\rho\sigma, \rho^3\sigma\} * \{e, \rho, \rho^2, \rho^3\} = Y_1^4. \end{aligned}$$

Similarly we calculte the others elements, so we obtained

$$C_{0,1} = X_0^2 * Y_1^4 = Y_1^4,\ C_{1,1} = X_1^2 * Y_1^4 = Y_1^4,\ C_{2,1} = X_2^2 * Y_1^4 = Y_0^4,\ C_{3,1} = X_3^2 * Y_1^4 = Y_0^4.$$

So, we have

$$\mathcal{G}_2^2 \circ \mathcal{G}_1^4 = \mathcal{G}_1^4. \tag{3}$$

Now, we analyzed the composion

$$\mathcal{G}_1^4 \circ \mathcal{G}_2^2 = \{C'_{0,0},\ C'_{0,1},\ C'_{0,2},\ C'_{0,3},\ C'_{1,0},\ C'_{1,1},\ C'_{1,2},\ C'_{1,3}\}.$$

According with (2) we noticed that $\rho^k\rho^p = \rho^{k+p} = \rho^p\rho^k = \rho^{p+k}$, for any $p, k \in \mathbb{N}^*$. So, we can conclude that

$$\begin{aligned} X_0^2 * Y_0^4 &= Y_0^4 * X_0^2 = Y_0^4 = C'_{0,0}, \\ X_1^2 * Y_0^4 &= Y_0^4 * X_1^2 = C'_{0,1} = Y_0^4, \\ C'_{0,2} &= Y_0^4 * X_2^2 = C'_{0,3} = Y_0^4 * X_3^2 = Y_1^4, \\ C'_{1,0} &= Y_1^4 * X_0^2 = C'_{1,1} = Y_1^4 * X_1^2 = Y_1^4, \\ C'_{1,2} &= Y_1^4 * X_2^2 = C'_{1,3} = Y_1^4 * X_3^2 = Y_0^4. \end{aligned}$$

Therefore, $\mathcal{G}_1^4 \circ \mathcal{G}_2^2 = \mathcal{G}_2^2 \circ \mathcal{G}_1^4 = \mathcal{G}_1^4$. Proceeding analogously we have

$$\mathcal{G}_1^4 \circ \mathcal{G}_4^4 = \mathcal{G}_4^1 \circ \mathcal{G}_1^4 = \mathcal{G}_1^4,\ \mathcal{G}_2^2 \circ \mathcal{G}_4^1 = \mathcal{G}_4^1 \circ \mathcal{G}_2^2 = \mathcal{G}_2^2.$$

In conclusion, $(\mathcal{G}_4, \circ)$is a commutative hyperstructure. □

**Remark 1.** *The elements of hyperstructure $(\mathcal{G}_4, \circ)$ satisfy the following equality*

$$\mathcal{G}_{p_2}^{p_1} \circ \mathcal{G}_{p'_2}^{p'_1} = \mathcal{G}_{p'_2}^{p'_1} \circ \mathcal{G}_{p_2}^{p_1} = \mathcal{G}_{\gcd\{p_2, p'_2\}}^{lcm\{p_1, p'_1\}}, \tag{4}$$

*for any $p_1$, $p_2$, $p'_1$, $p'_2 \in \mathbb{N}^*$ such that $4 = p_1p_2 = p'_1p'_2$.*

**Notation 1.** *$lcm\{p_1, p'_1\}$ represents the least common multiple of numbers $p_1, p'_1$, and $gcd\{p_2, p'_2\}$ is the greatest common divisor of $p_2, p'_2$.*

**Proposition 2.** *The hyperstructure $(\mathcal{G}_4, \circ)$ is a semihypergroup, but not is a quasihypergroup.*

**Proof.** $(\mathcal{G}_4, \circ)$ is a semihypergroup if and only if the hyperoperation "$\circ$" is associative, i.e.

$$\left(\mathcal{G}_{p_2}^{p_1} \circ \mathcal{G}_{p_2'}^{p_1'}\right) \circ \mathcal{G}_{p_2''}^{p_1''} = \mathcal{G}_{p_2}^{p_1} \circ \left(\mathcal{G}_{p_2'}^{p_1'} \circ \mathcal{G}_{p_2''}^{p_1''}\right),$$

for any $p_1, p_2, p_1', p_2', p_1'', p_2'' \in \mathbb{N}^*$ such that $4 = p_1 p_2 = p_1' p_2' = p_1'' p_2''$. We use the relation (4), and properties of gcd, respectively lcm:

$$\begin{aligned} \gcd\{\gcd\{p_2, p_2'\}, p_2''\} &= \gcd\{p_2, \gcd\{p_2', p_2''\}\}; \\ lcm\{lcm\{p_1, p_1'\}, p_1''\} &= lcm\{p_1, lcm\{p_1', p_1''\}\}. \end{aligned}$$

Therefore

$$\begin{aligned} \left(\mathcal{G}_{p_2}^{p_1} \circ \mathcal{G}_{p_2'}^{p_1'}\right) \circ \mathcal{G}_{p_2''}^{p_1''} &= \mathcal{G}_{\gcd\{p_2,p_2'\}}^{lcm\{p_1,p_1'\}} \circ \mathcal{G}_{p_2''}^{p_1''} = \mathcal{G}_{\gcd\{\gcd\{p_2,p_2'\},p_2''\}}^{lcm\{lcm\{p_1,p_1'\},p_1''\}} = \\ &= \mathcal{G}_{\gcd\{p_2,\gcd\{p_2',p_2''\}\}}^{lcm\{p_1,lcm\{p_1',p_1''\}\}} = \mathcal{G}_{p_2}^{p_1} \circ \left(\mathcal{G}_{p_2'}^{p_1'} \circ \mathcal{G}_{p_2''}^{p_1''}\right). \end{aligned}$$

To prove that the semihypergroup $(\mathcal{G}_4, \circ)$ is not a quasihypergroup, means that the hyperoperation doesn't satisfies the reproductive law, which is the following

$$\mathcal{G}_{p_2}^{p_1} \circ \mathcal{G}_4 = \mathcal{G}_4 \circ \mathcal{G}_{p_2}^{p_1} = \mathcal{G}_4,$$

for any $p_1, p_2 \in \mathbb{N}^*$, such that $4 = p_1 p_2$. We notice that

$$\mathcal{G}_1^4 \circ \mathcal{G}_4 = \mathcal{G}_1^4 \circ \mathcal{G}_1^4 \cup \mathcal{G}_1^4 \circ \mathcal{G}_2^2 \cup \mathcal{G}_1^4 \circ \mathcal{G}_4^1 = \mathcal{G}_1^4 \neq \mathcal{G}_4.$$

So, the semihypergroup $(\mathcal{G}_4, \circ)$ is not a quasihypergroup. □

**Remark 2.** *The cardinality of the semihypergroup $|\mathcal{G}_4|$ coincides with the number divisors of four.*

## 4. Conclusions

In this paper, we presented a way to obtain commutative hyperstructure, starting with a nonabelian group. We construct the hyperstructure $(\mathcal{G}_n, \circ)$ as a composition of the HX-groups associated to dihedral group $D_n$, $n \in \mathbb{N}^*$, $n > 2$. In the section Results and Discussions, we analyze this new structure in the particular cases, $n = 4$. We noticed that we have a commutative semihypergroup $(\mathcal{G}_4, \circ)$ which is not a quasihypergroup. Also, we see that exists a connection between the composition of HX groups and the function $lcm$, $gcd$ associated to them.

**Funding:**

**Institutional Review Board Statement:**

**Informed Consent Statement:**

**Data Availability Statement:**

**Conflicts of Interest:** The author declares no conflict of interest.

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