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On the Solutions for a Class of Boundary Value Problems of Fractional Type Using Coincidence Degree Theory †

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Abstract: This work is devoted to the study of a class of fractional boundary value problems using (left) Caputo derivative, and with the particularity of being resonant, i.e., the associated homogeneous problem admits a nontrivial solution. Conditions to ensure the existence and uniqueness of solutions are presented. Using Mawhin's coincidence degree, it is shown that the problem under consideration admits solutions and applying Banach contraction principle, sufficient conditions are obtained for which the solution is unique.

Keywords: fractional boundary value problem; Caputo derivative; Mawhin's coincidence degree; Banach contraction principle

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1. Introduction

Although Fractional Calculus has its origins in 1695, with a letter between Leibniz and L'Hôpital, it is only in the last decades that scientific interest in this area of mathematics has become evident. This is largely due to its many applications in some fields of engineering, biology, physics and mechanics (cf. [1–3]).

Obtaining analytical solutions in these type of problems is a very difficult issue. In this sense, there is a need to study if the solutions exist or not and if so, to analyse if there is uniqueness or not. Several methods are identified in the literature, from fixed point theorems, integral inequalities, the coincidence degree of Mawhin, etc. (see e.g., [4–13]).

Continuing the results presented in [14], in this paper, it is considered a class of boundary value problem of fractional order with (left) Caputo fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{a+}x(t) - f(t, x(t), x'(t), x''(t)) = 0, \quad t \in [a,b], \\ x(a) - \beta x'(a) = 0, \;\; x'(a) = x'(b) = \vartheta, \;\; x''(a) = 0, \end{cases} \tag{1}$$

where $\beta, \vartheta \in \mathbb{R}$, $0 \leq a < b$, $2 < \alpha < 3$ and $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. To obtain sufficient conditions to ensure the existence of solutions, the coincidence degree due to Mawhin is applied. The problem can be transformed into an equation of type $Lx = Nx$, with L being a linear operator between Banach spaces and N being the nonlinear part. Under that choice of boundary conditions, L is a not invertible (Ker$L \geq 1$), i.e., the problem is resonant. A more detailed approach can be found in [14]. Once the existence of solutions is guaranteed, we will proceed to the study of uniqueness.

2. Auxiliary Material and Methods

In this section, some essential definitions and methods are presented.

Definition 1. *The (left) Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function x is defined by*

$$I^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \mathrm{d}s,$$

admitting that the right-hand side is pointwise defined on (a, ∞), and with Γ being Euler Gamma function ($\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t,\ \alpha > 0$).

Definition 2. *The (left) Caputo fractional derivative of order $\alpha > 0$ of a continuous function x is defined by*

$$^{C}\mathcal{D}^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

with the right-hand side being pointwise defined on (a, ∞), and $n-1 < \alpha < n$, $n \in \mathbb{N}$.

Lemma 1 ([1]). *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$. If $x \in C^{n-1}([a,b])$, then it holds:*

$$(I^{\alpha}_{a+}\, {}^{C}\mathcal{D}^{\alpha}_{a+}x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k. \tag{2}$$

The following lemma is of great importance in Functional Analysis, and an essential tool in the proof of uniqueness and solutions.

Theorem 1 (Banach contraction principle). *Let (X, d) be a Banach space and let $T : X \to X$ be a contraction operator on X. Then, T has a unique fixed point $x \in X$.*

Mawhin's Coincidence Theory

Let X and Y be two normed spaces. In order to present Mawhin's coincidence theory, let us recall an important concepts.

A linear operator $L : \mathrm{dom}L \subset X \to Y$ is said to be a Fredholm operator with Fredholm index zero if $\mathrm{Im}L$ is a closed subset of Y and $\dim \mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < \infty$. If L is a Fredholm and its Fredholm index is zero, then there exist continuous projectors $P : X \to X$, $Q : Y \to Y$ such that

$$\mathrm{Im}P = \mathrm{Ker}L, \quad \mathrm{Ker}Q = \mathrm{Im}L, \quad X = \mathrm{Ker}L \oplus \mathrm{Ker}P, \quad Y = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Moreover, $L|_{\mathrm{dom}\, L \cap \ker P} : \mathrm{dom}\, L \cap \ker P \to \mathrm{Im}\, L$ is an isomorphism.

Definition 3. *Let Λ be an open bounded subset of X with $\mathrm{dom}L \cap \Lambda \neq \emptyset$. It is said that mapping N is L-compact on $\overline{\Lambda}$ if $QN(\overline{\Lambda})$ is bounded and $K_p(I-Q)N : \overline{\Lambda} \to X$ is completely continuous.*

We can present now the Mahwin's Theorem, which allows us to study the existence of solutions for the equation $Lx = Nx$.

Theorem 2 ([6]). *Let $\Lambda \subset X$ be open and bounded. Admit L i a Fredholm operator with Fredholm index zero and $N(\overline{\Lambda})$ is L-compact. Suppose that:*

(i) $Lx \neq \lambda Nx$ *for every* $x \in \partial\Lambda \cap (\mathrm{dom}L \backslash \mathrm{Ker}L)$ *and* $\lambda \in (0,1)$;
(ii) $Nx \notin \mathrm{Im}\, L$ *for every* $x \in \mathrm{Ker}L \cap \partial\Lambda$;
(iii) $\deg(QN|_{\mathrm{Ker}L}, \Lambda \cap \mathrm{Ker}L, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\mathrm{Im}L = \mathrm{Ker}Q$.

Then the equation $Lx = Nx$ *has at least one solution in* $\mathrm{dom}\, L \cap \overline{\Lambda}$.

In what follows, let $X = C^2([a,b])$ with the habitual norm

$$\|x\|_{C^2} = \max_{t \in [a,b]} \{\|x\|_{\infty} + \|x'\|_{\infty} + \|x''\|_{\infty}\}$$

and $Y = C([a,b])$ with the norm $\|y\|_C = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$. It is known that X and Y, considered with such norms, are Banach spaces.

3. Main Results

In what follows, it is applied the method presented in the last section is used. To that purpose, consider the operator $L : \mathrm{dom}L \subset C^2([a,b]) \to C([a,b])$ defined by

$$(Lx)(t) = ({}^C\mathcal{D}^\alpha_{a+}x)(t), \quad t \in [a,b], \tag{3}$$

where

$$\mathrm{dom}L = \{x \in C^2([a,b]) : ({}^C\mathcal{D}^\alpha_{a+}x)(t) \in Y,\ x(a) = \beta x'(a),\ x'(a) = x'(b) = \vartheta,\ x''(a) = 0\}.$$

Let $N : C^2([a,b]) \to C([a,b])$ be the operator

$$(Nx)(t) = f(t, x(t), x'(t), x''(t)), \quad t \in [a,b]. \tag{4}$$

Thus, the fractional boundary value problem (1) can be rewritten in the form:

$$Lx = Nx, \quad x \in \mathrm{dom}L, \tag{5}$$

(cf. [14]). Note that applying $L^{-1} = I^\alpha_a$ to both members of Equation (5), using boundary conditions (presented in (1)) and applying Lemma 1, it yields that

$$x(t) = \beta\vartheta + \vartheta(t-a) + I^\alpha_a(Nx)(t). \tag{6}$$

With some computations, it permit us to conclude that

$$\begin{aligned} \mathrm{Ker}L &= \{x \in C^2([a,b]) : x(t) = \vartheta(t-a+\beta),\ t \in [a,b]\}, \\ \mathrm{Im}L &= \left\{y \in C([a,b]) : \int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s = 0\right\}. \end{aligned}$$

Moreover, it is proved in [14] that L is a Fredholm operator of index zero (cf. [14]), and the linear continuous projectors $P : C^2([a,b]) \to C^2([a,b])$ and $Q : C([a,b]) \to C([a,b])$ can be defined as

$$\begin{aligned} (Px)(t) &= \vartheta(t-a+\beta), \\ (Qy)(t) &= \frac{\alpha-1}{(b-a)^{\alpha-1}} \int_a^b (b-s)^{\alpha-2} y(s)\mathrm{d}s, \quad t \in [a,b]. \end{aligned}$$

In order to conclude that the problem under study admit solutions, assume the following assertions:

(H1) There exist nonnegative constants p_1, p_2, p_3 and q such that

$$|f(t,x,y,z)| \le p_1|x(t)| + p_2|y(t)| + p_3|z(t)| + q, \quad t \in [a,b].$$

for any $(x,y,z) \in \mathbb{R}^3$, and so that $\eta \cdot p^* < 1$, with

$$\eta = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)} \tag{7}$$

and $p^* = \max_{t\in[a,b]}\{p_1, p_2, p_3\}$.

(H2) There exists a positive constant R such that for $x \in \mathrm{dom}L$, if $|x'(t)| > R$ for all $t \in [a,b]$, then

$$\int_a^b (b-s)^{\alpha-2} f(s, x(s), x'(s), x''(s))\mathrm{d}s \ne 0.$$

(H3) There exists a constant $R^* > 0$ such that for $c_1 \in \mathbb{R}$, if $|c_1| > R^*$ for $t \in [a,b]$, either

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) > 0,\ t \in [a,b],$$

or

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) < 0,\ t \in [a,b].$$

Theorem 3 (cf. [14]). *Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, and suppose conditions (H1), (H2) and (H3) are verified. Then the class of fractional boundary value problems (1) admits, at least, one solution in $C^2([a,b])$.*

Proof. Under the hypothesis (H1)–(H3), it is proved that the sets

$$\begin{aligned}
\Lambda_1 &= \{x \in \mathrm{dom}L \backslash \mathrm{Ker}L : Lx = \lambda Nx,\ \ \lambda \in (0,1)\} \\
\Lambda_2 &= \{x \in \mathrm{Ker}L : Nx \in \mathrm{Im}L\} \\
\Lambda_3 &= \{x \in \mathrm{Ker}L : \pm\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}
\end{aligned}$$

are bounded (cf. [14], Lemmas 7–10). Therefore, consider Λ to be a bounded open subset of $C^2([a,b])$ such that $\bigcup_{i=1}^{3} \overline{\Lambda}_i \subset \Lambda$. We claim that N is L-compact on $\overline{\Lambda}$ (cf. [14], Lemma 5) and L is a Fredholm operator with index 0 (cf. [14], Lemma 4). From the boundedness of sets Λ_1–Λ_3, it follows that:

(a) $Lx \neq \lambda Nx$ for every $x \in \partial\Lambda \cap (\mathrm{dom}L \backslash \mathrm{Ker}L)$ and $\lambda \in (0,1)$;
(b) $Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Lambda$;
(c) Let $H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx$. We know that $H(x,\lambda) \neq 0$ for $x \in \mathrm{Ker}L \cap \partial\Lambda$. Thus, by homotopy property of degree, we get

$$\begin{aligned}
\deg(QN|_{\mathrm{Ker}L}, \Lambda \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot,0), \mathrm{Ker}L \cap \partial\Lambda, 0) \\
&= \deg(H(\cdot,1), \mathrm{Ker}L \cap \partial\Lambda, 0) \\
&= \deg(\pm I, \mathrm{Ker}L \cap \partial\Lambda, 0) \neq 0,
\end{aligned}$$

(I represents the identity operator).

From (a)–(c), according to Theorem 2, there exists, at least, one solution for the equation $Lx = Nx$ in $\mathrm{dom}L \cap \overline{\Lambda}$, which ensure the existence of solutions for the problem (1) in $C^2([a,b])$, concluding the proof. □

The next theorem establishes sufficient conditions for the uniqueness of solutions.

Theorem 4. *Suppose that assertions (H1)–(H3) are verified and admit that there exist nonnegative constants d_1, d_2 and d_3 such that*

$$|f(t,x,y,z) - f(t,\overline{x},\overline{y},\overline{z})| \leq d_1|x-\overline{x}| + d_2|y-\overline{y}| + d_3|z-\overline{z}|, \tag{8}$$

for every $t \in [a,b]$, $(x,y,z) \in \mathbb{R}^3$, $(\overline{x},\overline{y},\overline{z}) \in \mathbb{R}^3$. If

$$\eta \cdot d^* < 1 \tag{9}$$

with η as defined in (7) and $d^* = \max\{d_1, d_2, d_3\}$, then there exists a unique solution in $C^2([a.b])$ for the the class of boundary value problem (1) under study.

Proof. In order to prove the uniqueness of solution in $C^2([a,b])$, according to (6), consider the operator $T : C^2([a,b]) \to C^2([a,b])$ defined by

$$\begin{aligned}
(Tx)(t) &= \beta\vartheta + \vartheta(t-a) + I_a^{\alpha}(Nx)(t) \\
&= \beta\vartheta + \vartheta(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\big(f(s,x(s),x'(s),x''(s))\big)ds.
\end{aligned} \tag{10}$$

Let $B_R = \{x \in C^2(\mathbb{R}) : \|x\|_{C^2} \leq R\}$ and choose

$$R \geq \frac{(|\vartheta| + b - a + 1)|\vartheta| + \eta q}{1 - \eta p^*},$$

with $p^* = \max\{p_1, p_2, p_3\}$. Note that, according to (H1), $1 - \eta p^* > 1$. For $2 < \alpha < 3$, the operator T is continuous and twice differentiable. Moreover, taking into account condition (H1), it follows that

$$\begin{aligned}
|Tx(t)| &\leq |\vartheta|(|\beta| + t - a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s, x(s), x'(s), x''(s))| ds \\
&\leq |\vartheta|(|\beta| + b - a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \big(p^*(|x(s)| + |x'(s)| + |x''(s)|) + q\big) ds \\
&\leq |\vartheta|(|\beta| + b - a) + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} q + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} p^* \|x\|_{C^2}.
\end{aligned}$$

Additionally, $(Tx)'(t) = \vartheta + \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} f(s, x(s), x'(s), x''(s)) ds$. Thus,

$$\begin{aligned}
|(Tx)'(t)| &\leq |\vartheta| + \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} |f(s, x(s), x'(s), x''(s))| ds \\
&\leq |\vartheta| + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} q + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} p^* \|x\|_{C^2}.
\end{aligned}$$

Finally, $(Tx)''(t) = \frac{1}{\Gamma(\alpha-2)} \int_a^t (t-s)^{\alpha-3} f(s, x(s), x'(s), x''(s)) ds$, and

$$\begin{aligned}
|(Tx)''(t)| &\leq \frac{1}{\Gamma(\alpha-2)} \int_a^t (t-s)^{\alpha-3} |f(s, x(s), x'(s), x''(s))| ds \\
&\leq \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)} q + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)} p^* \|x\|_{C^2}.
\end{aligned}$$

Thus, we obtain that $\|Tx\|_{C^2} \leq (|\beta| + b - a + 1)|\vartheta| + \eta q + \eta p^* R \leq R$, which shows that $T(B_R) \subset B_R$.

Now, take $x, y \in C^2([a,b])$. For any $t \in [a,b]$, we have that

$$\|Tx - Ty\|_{C^2} = \max_{t \in [a,b]} \{\|Tx - Ty\|_\infty + \|Tx' - Ty'\|_\infty + \|Tx'' - Ty''\|_\infty\}.$$

where $(Ty)(t) = \beta\vartheta + \vartheta(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s, y(s), y'(s), y''(s))) ds$, for any $y \in C^2([a,b])$.

Applying now (8), we obtain that

$$\begin{aligned}
\|Tx - Ty\| \leq{}& \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s, x(s), x'(s), x''(s)) - f(s, y(s), y'(s), y''(s))| ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} |f(s, x(s), x'(s), x''(s)) - f(s, y(s), y'(s), y''(s))| ds \\
&+ \frac{1}{\Gamma(\alpha-2)} \int_a^t (t-s)^{\alpha-3} |f(s, x(s), x'(s), x''(s)) - f(s, y(s), y'(s), y''(s))| ds \\
\leq{}& \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} d^* \big(|x(s) - y(s)| + |x'(s) - y'(s)| + |x''(s) - y''(s)|\big) ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} d^* \big(|x(s) - y(s)| + |x'(s) - y'(s)| + |x''(s) - y''(s)|\big) ds \\
&+ \frac{1}{\Gamma(\alpha-2)} \int_a^t (t-s)^{\alpha-3} d^* \big(|x(s) - y(s)| + |x'(s) - y'(s)| + |x''(s) - y''(s)|\big) ds
\end{aligned}$$

$$\begin{aligned} &\leq \quad d^*\|x-y\|_{C^2}\left(\frac{\int_a^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)} + \frac{\int_a^t (t-s)^{\alpha-2}ds}{\Gamma(\alpha-1)} + \frac{\int_a^t (t-s)^{\alpha-3}}{\Gamma(\alpha-2)}\right) \\ &= \quad \eta d^*\|x-y\|_{C^2}. \end{aligned}$$

Since $\eta \cdot d^* < 1$, by Banach contraction principle, T has a unique fixed point which is the unique solution of the problem (1), and the proof is complete. □

4. Conclusions

In this work, we consider a class of nonlinear resonant boundary value problem with (left) fractional Caputo derivative of order $\alpha \in (2,3)$. Applying Mawhin's coincidence Theorem, we obtained conditions that guarantee the existence of solutions of the problem (1). Imposing a Lipschitz condition on the function f and an inequality, with Banach contraction principle, we proved the uniqueness of solution.

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