computer sciences and mathematics forum

MDPI

Proceeding Paper

Applications of (h, q)-TimeScale Calculus to the Solution of Partial Differential Equations †

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† Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.

Abstract: In this article we developed the idea of q_time scale calculus in quantum geometry. It includes the q_time scale integral operators and Δ_q_differentials. Its analysis the fundamental principles which follow the calculus of q_time scale comparing with the Leibnitz-Newton usual calculus and have few crucial consequences. The Δ_q-differential reduced method of transformationis proposed to work out on partial Δ_q-differential equations in time scale. With easily computable coefficients the result is calculated in the version of a power series which is convergent. It is also illustrated the performance and effectiveness of the proposed procedure and applying Matlab software for calculation with the support of some fascinating examples. It changes when $\sigma(t) = t$ and $q = 1$, then the solution merges with usual calculus for the mentioned initial value problem. The finding of the present work is that the Δ_q_differential transformation reduced method is convenient and efficient.

Keywords: Δ_q-differential; q_time scale; q-Integral operators; Δ_q-differential reduced transform method; partial differential equation

1. Introduction

In sense of mathematical objects each and every theory of physics is articulated.

$$ih = [p, v] \tag{1}$$

Therefore, it is important to launch a number of formulas to frame any physical objects and concepts towards mathematical objectives where we study to epitomize them. As in classical mechanics, many times this function is appeared, in consequences for many theories, like mechanics of quantum, the mathematical things rarely revealing. This study focused mainly the closed quantum systems which consist of intrinsic components like states, observables, measurements, and evolution. Quantum geometry which dates back to the early days of quantum mechanics is characterized by Heisenberg's commutation relations [2,3]

These relations indicate that the classical phase space geometry is lost when position and momentum coordinates fail to commute. This leads to a non-commutative geometric space that is distinct from algebraic geometry, where the spaces are affine schemes built on a correspondence between spaces and commutative algebras. The Gelfand-Naimark theorem [8] provides a closer connection to differential geometry, associating spaces with topological spaces and commutative C*-algebras. Recent work has shown that non-commutative geometry in quantum geometry is intimately linked to delta q-deformed calculus, which is a generalization of quantum calculus. Our goal is to use Δ_q-calculus results to study non-commutative differential equations, specifically by employing the reduced

Citation: Abbas, M.A.; Muhammad, G.; Ali, H. Applications of (h, q)-TimeScale Calculus to the Solution of Partial Differential Equations. *Comput. Sci. Math. Forum* **2023**, *6*, x. https://doi.org/10.3390/xxxxx

Academic Editor(s):

Published: 28 April 2023

q delta-differential transform method to solve partial Δ_q-differential equations. Furthermore, we introduce that this algebra of operator [4–6] bring in shape some types of geometric space of non-commutative. On contrary algebraic geometry [7,8], that is established on a mapping between *commutative algebras and spaces*, this mapping in specific link with a part of space, the functions algebra on it, and then in a completely algebraic structure geometric concepts are described. This rule is the top logical initiative for general geometry like geometry of quantum. At the same time for, they are affine schemes spaces, for algebraic geometry. Noncommutative differential equations in time sca calculus have important applications in various areas of mathematics and physics, including quantum groups, quantum field theory, and statistical mechanics.

In quantum geometry Maliki et al. in [9] studied the concepts of deformed q_calculus. Here, they demonstrated how the q_calculus, an expanded version of the Leibnitz and Newton standard calculus, and the mathematical discipline of invariant geometry are intimately related. In this study, we review a few results from the q_time scale calculus that will aid in our observation of invariant differential equations. In particular, we will use the q_Delta-differential transform reduced method (qDDRTM) to examine partial q_Delta differential equations.

1.1. Operator of the q_Delta Differential

For $1 < q \in < R$, we establish the delta q_derivative Δ_q as;

$$\Delta_q f(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \tag{2}$$

Note that $\Delta_q \to f'(t) \equiv \frac{d}{dt}$, as q tends to 1

We assume the following supporting notable points.

(a) F.H. Jackson [10] discussed the modified version of q_derivative and its several consequences in 20th century.

(b) The functions those do not have 0 in their definition domain the Δ_q_derivative can be calculated for those functions. As q is at 1, it decreases to a common derivative.

(c) It can be checked easily that the Δ_q_operator is linear operator, i.e.,

$$\begin{aligned} &\text{(i) } \Delta_q(f+g) = \Delta_q f + \Delta_q g \\ &\text{(ii) } \Delta_q(\lambda f) = \lambda \Delta_q f \end{aligned} \tag{3}$$

1.2. The q_Δ Derivative of Few Transcendental Mappings and Non Commutative Concept

Maliki et al. in [9] discussed noncommutative differential equation in q calculus is the q-difference equation

$$D_q f(x) = \frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \text{ where } x \neq 0, 0 < q < 1$$

$f(qx) - f(x) = \dfrac{\frac{d_q}{d_q x} f(x)(q-1)}{q}$ where q is a deformation parameter that determines the degree of noncommutativity in the calculus.

$$f(\sigma(t)) - f(t) = \frac{\Delta_q f(t) \mu(t)}{q}$$

The noncommutativity of the q delta-derivative makes it more challenging to solve, but methods such as the q delta-differential transform method can be used to obtain solutions.

Following the method of calculating the Δ_q_derivative (non commutative in time scales calculus) by the first principles, we now get salient features for the Δ_q_operatorof the below mentioned mappings like e^t and Sint.

q_Delta operator of the Function

$$h(\mathrm{t}) = \sin t$$

By definition, we have

$$\Delta_{\mathrm{q}} Sin(t) = \frac{\sin(\sigma(t)) - \sin t}{\mu(t)}$$

$$\Delta_{\mathrm{q}}(Sint) = \frac{\left(\sigma(t) - \frac{1}{3!}(\sigma(t))^3 + \frac{1}{5!}(\sigma(t))^5 - \frac{1}{7!}(\sigma(t))^7 + \cdots\cdots\right) - \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\cdots\right)}{\mu(t)} \quad (4)$$

Note: by setting $\sigma(t) = qt$in $\Delta_{\mathrm{q}}(Sin\ t)$we obtain the results of quantum calculus.
By using theresult:
When $\sigma(t) = t$ in (4), we get the standard derivative of sine *function*, i.e.,

$$\frac{d}{dt}(\sin t) = 1 - \frac{1}{3!}t^2(3) + \frac{1}{5!}t^4(5) - \frac{1}{7!}t^6(7) + \cdots\cdots$$
$$= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots\cdots = cos\,t. \quad (5)$$

1.3. q_Time Scale Factorials and q_Timescale Numbers

Basically, we adopt the notations and symbols in [11]. Thus$\mathbb{Z}^+$ denoted the set of integers which is positive. More, *M*denotes a field which has 0 characteristic throughout this research article and M(q) represents the rational functions field in one parameter *q* over N(q). In the *q*_deformed setting N(q) is our ground field, while *N*is the ground field in the standard setting. We define the *q*_binomials, *q*_integers and*q*_factorials respectively as follows:

1. $[\![p]\!]_q = \frac{q^t - 1}{q-1} = \sum_{j=0}^{p-1} q^j$ (6)
2. $[\![p]\!]_q! = [\![p]\!]_q \times [\![p-1]\!]_q \times [\![p-2]\!]_q \times \cdots\cdots \times [\![3]\!]_q \times [\![2]\!]_q \times [\![1]\!]_q$; where $[\![0]\!]_q! = 1$ (7)
3. $\left\{ {p \atop r} \right\}_q = \frac{[\![p]\!]_{q!}}{![\![p-r]\!]_q!\,[\![r]\!]_q}$, ∀ p, r∈ N_0, p ≥ r (8)

We have an example g (t) = t^p, then

$$\Delta_q g(t) = \sigma(t)^{p-1} + t\sigma(t)^{p-2} + \sigma(t)^{p-3}t^2 + \cdots\cdots\cdots\cdots \mathrm{t}^{p-1}$$

Setting $\sigma(t) = t$, we have classical derivative $\frac{d}{dt}g(t)$= p t^{p-1}
And in q_calculus it becomes:

$$\frac{d}{d_q t}(g(t)) = [\![p]\!]_q\ \mathrm{t}^{p-1};\text{where we consider } \sigma(t) = qt \quad (9)$$

Remarks

The characteristics and proofs of the *q*_factorials, *q*_integers and *q*_binomials are discussed in [9]. Now we have the below results on the Δ_q_operator. For $q \neq 1 \in \mathbb{R}$, and with Δ_q is defined here.

$$(1)\ \Delta_q g(t) = \sum_{p=0}^{\infty} \frac{(\mu(t))^p}{(1+r)!} \frac{d^{p+1}}{dt^{p+1}} g(t) \quad (10)$$

$$(2)\ \Delta_q^p g(t) = \frac{q^{(p-1)/2}}{\sigma(t)^p (q-1)^p} \sum_{r=0}^{p} \left\{ {p \atop r} \right\}_q f(q^{p-1-j}\sigma(t))(-1)^j q^{\frac{r(r-1)}{2}} \quad (11)$$

(3) $\Delta_q\{w(t)v(t)\} = v(t)\Delta_q w(t) + w\big(\sigma(t)\big)\Delta_q v(t)$ (12)

(4) $\Delta_q\left\{\frac{w(t)}{v(t)}\right\} = \frac{v(\sigma(t))\Delta_q w(t) - w(\sigma(t))\Delta_q v(t)}{v(t)v(\sigma(t))}$ (13)

1.4. Partial Δ_q-Derivative of Multivariable Function in Time Scale

We define the continuous multivariable real valued function and partial Δ_q_derivative of a function $g(t_1, t_2, \ldots, t_n)$with respect to a variable $t_i$by;

$$\Delta_{q,t_j} g(t) = \frac{(\partial_{q,j} g)(t) - g(t)}{(1-q)t_j} \tag{14}$$

$$\left[\Delta_{q,t_j} g(t)\right]_{t_j=0} = \lim_{t_j \to 0} \Delta_{q,t_j} g(t) \tag{15}$$

where $\mathrm{t} = (\mathrm{t}_1, \mathrm{t}_2, \cdots\cdots, \mathrm{t}_\mathrm{p})$

And $(\partial_{q,j} g)(t) = g\big(\mathrm{t}_1, \mathrm{t}_2, \cdots\cdots, \sigma(t_j) \cdots\cdots\cdots t_\mathrm{n}\big)$

For the j^{th}order $\Delta_q$order derivative subsequently we adopt the identity with respect to t^j.

The solution of partial Δ_q_differential equations using the innovative q_differential transform approach presented in [13] is currently our main goal.

(1) Δ_q_Differential reduced Transformation Method Considering that all Δ_q_differentials of v(t, x) exist in a region where x = a, we shall let

$$V_j(t) = \frac{1}{[\![j]\!]_q!}\left[\frac{\partial_q^j}{\Delta_q x^j} v(t,x)\right]_{t=a} \tag{16}$$

where $W_j(x)$ is the transformed spectrum function of x-dimensional. Resultantly, the uppercase $W_j(t)$ represents for the transformed mapping. While mapping of lower case $v(t,x)$ shows the original function. Now we have the below important definition.

Definition. *The inverse transform of Δ_q_differential of $W_j(t)$ is defined by;*

$$v(t,x) = \sum_{j=0}^{\infty} W_K(t)(x-c)^{(j)} \tag{17}$$

putting Equations (16) in Equation (17) we get:

$$W_j(t) = \sum_{j=0}^{\infty} W_K(t) = \frac{1}{[\![j]\!]_q!}\left[\frac{\partial_q^j}{\Delta_q x^k} v(t,x)\right]_{t=a} (x-c)^{(j)}$$

In the coming theorems, we let c = 0 such that $(x-c)^{(j)}$=$(x-0)^{(j)}$=$(t)^{(j)}$.

We can construct the below mentioned fact from the linearity of the Δ_q-derivative, given $z(t,x) = \beta v(t,x) \pm u(t,x)$ then $Z_j(t) = W_j(t) \pm U_j$. We have the following important theoremas β being a constant.

Theorem. *Given $z(t,x)$= $t^r x^p$ then $Z_j(t) = t^r \delta(j-p)$ where*

$$\delta(j) = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0 \end{cases} \tag{18}$$

Proof. From definition (20), we have;

$$Z_j(t) = \frac{1}{[[\![j]\!]]_q!}\left[\frac{\partial_q^j(t^r x^p)}{\Delta_q x^j} v(t,x)\right]_{x=0} = \frac{t^r}{[[\![p]\!]]_q!}\left[\frac{\partial_q^j(t^m x^p)}{\Delta_q x^j} v(t,x)\right]_{x=0}$$

$$\begin{cases} t^r.\dfrac{[\![p]\!]_{q}!}{[\![p]\!]_q!} = t^r, \quad j = p \\ t^r.\dfrac{[\![p_q]\!].[\![(p-1)]\!]_q \dots [\![(p-j+1)]\!]_q}{[\![p]\!]_q!} \quad x^{p-j}\big|_{x=0}, \quad j \neq 0 \\ t^r.0 = 0. \quad j > p \end{cases} \tag{19}$$

$$= t\delta(j-p)$$

Theorem. *Given* $z(t,x) = \frac{\partial_q}{\Delta_q t}\mathrm{v}(t,x)$ *then* $Z_j(t) = \frac{\Delta_q}{\partial_q t}\mathrm{V}(t)$.

Proof.

$$Z_j(t) = \frac{1}{[\![j]\!]_q!}\left[\frac{\partial_q^j}{\Delta_q x^j}\left(\frac{\partial_q}{\Delta_q{}^t}v(t,x)\right)\right] = \frac{1}{[\![j]\!]_q!}\left[\frac{\partial_q}{\Delta_q{}^x}\left(\frac{\partial_q^j}{\Delta_q x^j}v(t,x)\right)\right]_{x=0} \tag{20}$$

$$\frac{\partial_q}{\Delta_q{}^t}\frac{1}{[\![j]\!]_q!}\left[\frac{1}{[\![j]\!]_q!}\left(\frac{\partial_q^j}{\Delta_q x^j}v(t,x)\right)\right]_{t=0} = \frac{\partial_q}{\Delta_q{}^t}V_j(t)$$

Theorem. *Given* $z(t,x) = \frac{\partial_q}{\Delta_q t}(\mathrm{v}(t,x))$, *then*

$$Z_j(t) = [\![j+1]\!]_q[\![j+2]\!]_q \cdots\cdots [\![j+k]\!]_q V_{j+k}(t) \tag{21}$$

Proof.

$$Z_j(t) = \frac{1}{[\![j]\!]_q!}\left[\frac{\partial_q^j}{\Delta_q x^j}\left(\frac{\partial_q^k}{\Delta_q x^r}v(t,x)\right)\right]$$

$$= \frac{[\![j+k]\!]_q!}{[\![j]\!]_q!}\frac{1}{[\![j+k]\!]_q!}\left[\left(\frac{\partial_q^{j+k}}{\Delta_q x^{j+k}}v(t,x)\right)\right]_{x=0}$$

$$= [\![j+1]\!]_q[\![j+2]\!]_q \cdots\cdots [\![j+k]\!]_q V_{j+k}(t)$$

Example.

$$\frac{\partial_q}{\Delta_q{}^x}v(t,x) = v^2(t,x) + \frac{\partial_q}{\Delta_q{}^t}v(t,x), v(t,0) = 1 + 3t \tag{22}$$

Using the reduced q_differetial transform method of the given partial Δ_q_differential equation, we obtain:

$$[\![j+1]\!]_q V_{j+1}(t) = \textstyle\sum_{p=0}^{j} V_{j-p}(t)V_j(t) + \frac{\partial_q}{\Delta_q{}^x}V_j(t), \tag{23}$$

The given initial condition forms.

$$V_O(t) = v(t,o) = 1 + 3t.$$

Initiating with j = 0, the values of the function $V_j(t)$ are calculated successively as given below;

$$[\![1]\!]_q V_1(t) = V_o(t)V_o(t) + \frac{\partial_q}{\Delta_q{}^t}V_o(t)$$

$$= (1+3t)^2 + \frac{\partial_q}{\Delta_q{}^t} V_o(1+3t) = (1+3t)^2 + 3 \tag{29}$$

$$V_1(t) = 4 + 6t + 9t^2$$

where j = 1, we have

$$\left(\frac{q^2-1}{q-1}\right) V_2(t) = 2V_o(t)V_1(t) + \frac{\partial_q}{\Delta_q{}^t} V_1(= 2(1+3t)(4+6t+9t^2)$$

$$+ \frac{\partial_q}{\Delta_q{}^t} U_o(4+6t+9t^2)$$

$$\therefore V_2(t) = \frac{14 + 9(5t + \sigma(t)) + 54t^2 + 54t^3}{1+q} \tag{24}$$

Below are the same method, it is simple to calculate an expression for 2 = j.

Normally, we need solution of the partial Δ_q_differential equation to be:

$$\mathrm{v(t,x)} = 1 + 3\mathrm{t} + (4 + 6t + 9t^2)\mathrm{x} + \left(\frac{14 + 9(\sigma(t) + 5t) + 54t^2 + 54t^3}{1+q}\right)\mathrm{x}^2 + \tag{25}$$

Let’s now determine the classical form of the partial q_Delta differential equation provided, i.e.,

$$\frac{\partial}{\Delta t} v(t,x) = v^2(t,x) + \frac{\partial}{\Delta t} v(t,x),$$

$$v(t,0) = 1 + 3t \tag{26}$$

By the characteristics method the above first order partial differential equation of Quasilinear can be solved easily. The auxiliary associated equations are;

$$\frac{dx}{1} = \frac{dv}{v^2} = -\frac{dt}{1} \tag{27}$$

These provide two potential integrals that are provided by;

$$\mathrm{x} + \mathrm{t} = \mathrm{a}_1 \text{ and } \frac{1}{\mathrm{v}} + \mathrm{x} = \mathrm{a}_2 \tag{28}$$

where, a_1, a_2 are considered as constants of integration which are arbitrary. Using the initial condition, we have at x = 0, v = 1 + 3t. Hencet = a and $\frac{1}{1+3\mathrm{t}} = \mathrm{a}_2$.

It then gives that

$$\frac{1}{1+3\mathrm{a}_1} = \mathrm{a}_2.$$

Consequently, the necessary solution is

$$\mathrm{x} + \frac{1}{\mathrm{t}} = \frac{1}{1 + 3(\mathrm{x}+\mathrm{t})} \text{ or } \mathrm{v(x,t)} = \frac{1 + 3(\mathrm{x}+\mathrm{t})}{1 - \mathrm{x}(1+3\mathrm{t}) - 3\mathrm{x}^2} \tag{29}$$

Using MatLab, a software for numerical solution and adding powers of x to the formula for v results in the following.

$$\mathrm{v(x,t)} = (1 + 3\mathrm{t}) + (4 + 6\mathrm{t} + 9\mathrm{t}^2)\mathrm{x}$$

$$+ (7 + 27\mathrm{t}^2 + 27\mathrm{t}^3)\mathrm{x}^2 + \cdots \tag{30}$$

Here we have an interesting observation that when we put $\sigma(t) = t$ and $q = 1$ in (30) we successfully arrive at the conventional PDE’s solution.

1.5. Conclusions

The concept of the q_time scale calculus in quantum geometry is described in this research article. For this purpose, we include discussion of the principlefrom q_calculus comparing with the usual-Leibnitz and Newton calculus. Our basic aim is to obtainthe consequences acquired to deal partial Δ_q_differential equations. For this goal we initiated the concept of the Δ_q_differential reduced method of transformation that leads convergent solution of power serieswith easily computable sections. We were able to demonstrate the effectiveness and convenience of the proposed iteration technique using a few cases. It merges to standard form solution with initial value problems that when $q = 1$ and $\sigma(t) = t$. Deduce conclusion is thatq_time scale calculus is invariant, that is non-commutative calculus which coincides the Leibnitz-Newton standard calculus. This work introduced and generalized the qDDTM to work on partial q_differential equations, which represent non-commutative spaces form of some dynamics.

Data Availability Statement: The data used in this article will be provided upon the reasonable request.

Acknowledgments: The authors are grateful to the research team for their dedication and hardworking.

Conflicts of Interest:

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