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Existence and Uniqueness of a Solution of a Wentzell's Problem with Nonlinear Delays †

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Abstract: In this work, we study the existence and the uniqueness of the solution of a wave equation with dynamic Wentztell type boundary conditions on a part of the boundary Γ_1 of the domain Ω with nonlinear delays in nonlinear dampings in Ω and on Γ_1, using Faedo-Galerkin's method.

Keywords: wave equation; wentzell boundary conditions; nonlinear dampings; nonlinear delays; Faedo-Galerkin's method

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1. Introduction

We consider the following coupled system wave/Wentzell :

$$\begin{cases} u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(u_t(t-\tau)) = 0, & \text{in } \Omega \times (0,\infty), \\ v_{tt} + \partial_\nu u - \Delta_T v + \mu_1' g_2(v_t) + \mu_2' g_2(v_t(t-\tau)) = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ u = v, & \text{on } \Gamma \times (0,\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ (u(0), v(0)) = (u_0, v_0), & \text{in } \Omega \times \Gamma, \\ (u_t(0), v_t(0)) = (u_1, v_1), & \text{in } \Omega \times \Gamma, \\ u_t(x, t-\tau) = f_{0_1}(x, t-\tau), & \text{in } \Omega \times (0,\tau), \\ v_t(x, t-\tau) = f_{0_2}(x, t-\tau), & \text{on } \Gamma_1 \times (0,\tau), \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $(n \geq 2)$, with smooth boundary $\Gamma = \partial\Omega$, divided into two closed and disjoint subsets Γ_0 and Γ_1, such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $\Gamma_0 \cup \Gamma_1 = \Gamma$. We denote by ∇_T the tangential gradient on Γ, by Δ_T the tangential Laplacian on Γ and by ∂_ν the normal derivative where ν represents the unit outward normal to Γ.

μ_1, μ_2, μ_1' and μ_2' are positive real numbers, the two functions $g_1(u_t(t-\tau))$ and $g_2(v_t(t-\tau))$ describe the delays on the nonlinear frictional dissipations $g_1(u_t)$ and $g_2(v_t)$, on Ω and Γ_1, respectively, $\tau > 0$ is a time delay and u_0, v_0, u_1, v_1, f_{0_1} and f_{0_2} are the initial data in some suitable (Sobolev) function spaces.

Throughout history, the wave equation has known a great deal of work.

In our work, we are particularly interested in the wave equation with boundary conditions of the Wentzell type, which are characterized by the presence of differential operators $(\Delta_T u)$ of the same order as the main operator.

These problems are involved in the modeling of many phenomena: mechanical like elasticity, physics such as diffusion processes or wave propagation.

Wentzell's conditions are obtained by asymptotic methods from transmission problems, (see Lemrabet. K [1]).

The following condition :

$$\partial_\nu u - \Delta_T u = g, \quad \text{on } \Gamma$$

for this equation

$$-\triangle u + u = f, \quad \text{in}\,\Omega$$

was first introduced by Wentzell (Ventcel) in 1959, (see [2]), for diffusion processes. It models the heat exchange of the body Ω with the surrounding environment in the presence of a thin film, very good conductor, on the surface of the body.

Delay is the property of a physical system by which the response to an applied force is retarded in its effect. Whenever material, information, or energy is physically transmitted from one place to another, there is a delay present, a delay in the law of feedback modeling mechanical shift over time.

Delays so often occur in many: physical problems, chemical, biological and economic phenomena.

The system (1) describes vibrations of a flexible body with a thin boundary layer of high rigidity on its boundary Γ_1.

Our goal is to show that this problem is well posed, that there is existence and uniqueness of a solution.

1.1. *Assumptions on the Damping and Delay Functions g_i for $i = 1, 2$:*

We pose the following assumptions on the damping and delay functions:

(**A1**) $g_i : \mathbb{R} \longrightarrow \mathbb{R}$ is an odd non decreasing function of the class $\mathcal{C}^1(\mathbb{R})$ such that there exist r (sufficiently small), c_i, C_i, c, α_1 and $\alpha_2 > 0$ for $i = 1, 2$, and a convex, increasing function $H : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of the class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(]0, \infty[)$ satisfying :

$H(0) = 0$ and H linear on $[0, r]$ or $(H'(0) = 0$ and $H'' > 0$ on $]0, r])$, such that

$$c_i|s| \leq |g_i(s)| \leq C_i|s| \quad \text{if } |s| \geq r, \tag{2}$$

$$s^2 + g_i^2(s) \leq H^{-1}(sg_i(s)) \quad \text{if } |s| \leq r, \tag{3}$$

$$\left|g_i'(s)\right| \leq c, \tag{4}$$

$$\alpha_1 s g_i(s) \leq G_i(s) \leq \alpha_2 s g_i(s), \tag{5}$$

where

$$G_i(s) = \int_0^s g_i(y)dy.$$

(**A2**) $\alpha_2\mu_2 < \alpha_1\mu_1$ and $\alpha_2\mu_2' < \alpha_1\mu_1'$.

1.2. *Transformation of Problem (1)*

Now, as in the work of [3], we introduce the new variables:

$$\begin{cases} z_1(x, \rho, t) = u_t(x, t - \rho\tau), & x \in \Omega, \ \rho \in (0, 1), \ t > 0, \\ z_2(x, \rho, t) = v_t(x, t - \rho\tau), & x \in \Gamma_1, \ \rho \in (0, 1), \ t > 0, \end{cases}$$

where $\tau > 0$ is a time delay.

Then, we have

$$\begin{cases} \tau(z_1)_t(x, \rho, t) + (z_1)_\rho(x, \rho, t) = 0, & \text{on } \Omega \times (0, 1) \times (0, \infty), \\ \tau(z_2)_t(x, \rho, t) + (z_2)_\rho(x, \rho, t) = 0, & \text{on } \Gamma_1 \times (0, 1) \times (0, \infty), \end{cases}$$

where $(z_i)_t = \frac{\partial z_i}{\partial t}$ and $(z_i)_\rho = \frac{\partial z_i}{\partial \rho}$, for $i = 1, 2$.

Therefore, problem (1) is equivalent to

$$
\begin{cases}
u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(z_1(x,1,t)) = 0, & \text{in} \quad \Omega \times (0,\infty), \\
v_{tt} + \partial_\nu u - \Delta_T v + \mu_1' g_2(v_t) + \mu_2' g_2(z_2(x,1,t)) = 0, & \text{on} \quad \Gamma_1 \times (0,\infty), \\
\tau (z_1)_t(x,\rho,t) + (z_1)_\rho(x,\rho,t) = 0, & \text{in} \quad \Omega \times (0,1) \times (0,\infty), \\
\tau (z_2)_t(x,\rho,t) + (z_2)_\rho(x,\rho,t) = 0, & \text{on} \quad \Gamma_1 \times (0,1) \times (0,\infty), \\
u = v, & \text{on} \quad \Gamma \times (0,\infty), \\
u = 0, & \text{on} \quad \Gamma_0 \times (0,\infty), \\
z_1(x,0,t) = u_t(x,t), & \text{in} \quad \Omega \times (0,\infty), \\
z_2(x,0,t) = v_t(x,t), & \text{on} \quad \Gamma_1 \times (0,\infty), \\
(u(0), v(0)) = (u_0, v_0), & \text{in} \quad \Omega \times \Gamma, \\
(u_t(0), v_t(0)) = (u_1, v_1), & \text{in} \quad \Omega \times \Gamma, \\
z_1(x,\rho,0) = f_{0_1}(x,-\rho\tau), & \text{in} \quad \Omega \times (0,1), \\
z_2(x,\rho,0) = f_{0_2}(x,-\rho\tau), & \text{on} \quad \Gamma_1 \times (0,1).
\end{cases}
\tag{6}
$$

1.3. Energy of System (6)

Let ξ and ζ be strictly positive constants, such that

$$
\tau \frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau \frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}, \tag{7}
$$

$$
\tau \frac{\mu_2'(1-\alpha_1)}{\alpha_1} < \zeta < \tau \frac{\mu_1' - \alpha_2\mu_2'}{\alpha_2}. \tag{8}
$$

We define the energy associated to the solution of problem (6) by

$$
\begin{aligned}
E(t) = {} & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|v_t\|^2_{\Gamma_1} + \frac{1}{2}\|\nabla_T v\|^2_{\Gamma_1} \\
& + \xi \int_\Omega \left(\int_0^1 G_1(z_1(x,\rho,t))d\rho \right) dx + \zeta \int_{\Gamma_1} \left(\int_0^1 G_2(z_2(x,\rho,t))d\rho \right) d\sigma,
\end{aligned}
\tag{9}
$$

where $\|.\| = (.,.)^{\frac{1}{2}}$ and $\|.\|_{\Gamma_1} = (.,.)^{\frac{1}{2}}_{\Gamma_1}$, (the norms associated with the inner products in $L^2(\Omega)$ and $L^2(\Gamma_1)$, respectively).

1.4. Energy Decay

We have the following lemma on the dissipation of energy $E(t)$:

Lemma 1. *Let (u, v, z_1, z_2) be a solution of problem (6). Then, the energy functional defined by (9) satisfies*

$$
\begin{aligned}
E'(t) \quad \leq \quad & -a_1 \int_\Omega u_t g_1(u_t)dx - a_2 \int_{\Gamma_1} v_t g_2(v_t)d\sigma \\
& -a_3 \int_\Omega z_1(x,1,t)) g_1(z_1(x,1,t))dx \\
& -a_4 \int_{\Gamma_1} z_2(x,1,t)) g_2(z_2(x,1,t))d\sigma \\
\leq \quad & 0, \qquad\qquad \forall t \geq 0,
\end{aligned}
\tag{10}
$$

where $a_1 = \left(\mu_1 - \frac{\xi}{\tau}\alpha_2 - \mu_2\alpha_2\right)$, $a_2 = \left(\mu_1' - \frac{\zeta}{\tau}\alpha_2 - \mu_2'\alpha_2\right)$, $a_3 = \left(\alpha_1\frac{\xi}{\tau} - \mu_2(1-\alpha_1)\right)$ *and* $a_4 = \left(\alpha_1\frac{\zeta}{\tau} - \mu_2'(1-\alpha_1)\right)$.

For the proof of Lemma 1, we can see [4].

2. Main Result

We introduce the following set

$$H^1_{\Gamma_0}(\Omega) = \Big\{ u \in H^1(\Omega) \;/\; u|_{\Gamma_0} = 0 \Big\},$$

which is endowed with the Hilbert structure induced by $H^1(\Omega)$.

Then, we consider the canonical norms of $H^1_{\Gamma_0}(\Omega)$ and $H^1(\Gamma_1)$

$$\|u\|^2_{H^1_{\Gamma_0}(\Omega)} = \|\nabla u\|^2, \qquad \|v\|^2_{H^1(\Gamma_1)} = \|\nabla_T v\|^2_{\Gamma_1}.$$

Now, we state the following existence and uniqueness result:

Theorem 1. *Let $(u_0, u_1, v_0, v_1) \in [H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega)] \times [H^2(\Gamma_1) \times H^1(\Gamma_1)]$, $f_{0_1} \in H^1_{\Gamma_0}\big(\Omega; H^1(0,1)\big)$ and $f_{0_2} \in H^1\big(\Gamma_1; H^1(0,1)\big)$ satisfying the following compatibility condition:*

$$\begin{cases} \partial_\nu u_0 - \Delta_T v_0 + \mu_1' g_2(v_1) = 0, & on \quad \Gamma_1, \\ f_{0_1}(.,0) = u_t, & in \quad \Omega, \\ f_{0_2}(.,0) = v_t, & on \quad \Gamma_1. \end{cases} \tag{11}$$

Assume that $(\mathbf{A1})$ *and* $(\mathbf{A2})$ *hold, then problem (6) possesses a unique global weak solution verifying for $T > 0$:*

$$(u, u_t, u_{tt}) \in L^\infty(0,T; \Big[H^1_{\Gamma_0}(\Omega)\Big]^2 \times L^2(\Omega)),$$
$$(v, v_t, v_{tt}) \in L^\infty(0,T; \Big[H^1(\Gamma_1)\Big]^2 \times L^2(\Gamma_1)).$$

We shall give a proof of Theorem 1, by using the Faedo-Galerkin's approximation.

Proof of Theorem 1. Throughtout this proof, assume $(u_0, v_0) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times \big(H^2(\Gamma_1) \cap H^1(\Gamma_1)\big)$, $(u_1, v_1) \in H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1)$, $f_{0_1} \in H^1_{\Gamma_0}\big(\Omega; H^1(0,1)\big)$ and $f_{0_2} \in H^1 \big(\Gamma_1; H^1(0,1)\big)$.

For any $n \in \mathbb{N}$, we denote by U_n and V_n the two finite dimentional spaces defined by respectively $U_n = span\{w_1, w_2, ..., w_n\}$ and $V_n = span\{\widetilde{w}_1, \widetilde{w}_2, ..., \widetilde{w}_n\}$, where $\{w_i\}_{1 \le i \le n}$ and $\{\widetilde{w}_i\}_{1 \le i \le n}$ are basis in the spaces $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and $H^2(\Gamma_1) \cap H^1(\Gamma_1)$, respectively.

Now, define for $1 \le i \le n$ the sequences $\phi_i(x,\rho)$ and $\widetilde{\phi}_i(x,\rho)$ as follows:

$$\begin{cases} \phi_i(x,0) = w_i, \\ \widetilde{\phi}_i(x,0) = \widetilde{w}_i, \end{cases}$$

then, extend $\phi_i(x,0)$ by $\phi_i(x,\rho)$ over $L^2(\Omega \times (0,1))$ and $\widetilde{\phi}_i(x,0)$ by $\widetilde{\phi}_i(x,\rho)$ over $L^2(\Gamma_1 \times (0,1))$ and denote Z_n, $\widetilde{Z}_n$ the linear spaces generated by $\{\phi_1, \phi_2, ..., \phi_n\}$ and $\{\widetilde{\phi}_1, \widetilde{\phi}_2, ..., \widetilde{\phi}_n\}$, respectively.

Let us define the approximations u^n, v^n, z_1^n and z_2^n by

$$u^n(t) = \sum_{i=1}^n a_i^n(t) w_i, \; v^n(t) = \sum_{i=1}^n b_i^n(t) \widetilde{w}_i, \; z_1^n(t) = \sum_{i=1}^n c_i^n(t) \phi_i, \; z_2^n(t) = \sum_{i=1}^n d_i^n(t) \widetilde{\phi}_i,$$

where a_i^n, b_i^n, c_i^n and d_i^n are from the class $\mathcal{C}^2$ and determined by following differential equations:

$$\begin{aligned} &(u^n_{tt}, w_i) + (\nabla u^n, \nabla w_i) + \mu_1(g_1(u^n_t), w_i) + \mu_2(g_1(z_1^n(x,1,t)), w_i) \\ &+(v^n_{tt}, \widetilde{w}_i)_{\Gamma_1} + (\nabla_T v^n, \nabla_T \widetilde{w}_i)_{\Gamma_1} + \mu_1'(g_2(v^n_t), \widetilde{w}_i)_{\Gamma_1} \\ &+\mu_2'(g_2(z_2^n(x,1,t)), \widetilde{w}_i)_{\Gamma_1} = 0, \qquad 1 \le i \le n, \end{aligned} \tag{12}$$

$$\int_\Omega \int_0^1 (\tau z^n_{1_t} + z^n_{1_\rho})\phi_i d\rho dx = 0, \qquad 1 \le i \le n \tag{13}$$

and

$$\int_{\Gamma_1} \int_0^1 (\tau z^n_{2_t} + z^n_{2_\rho})\widetilde{\phi}_i d\rho d\sigma = 0, \qquad 1 \le i \le n, \tag{14}$$

with initial data:

$$\begin{cases} u^n(0) &= u^n_0 &= \sum_{i=1}^n a^n_i(0) w_i &\to u_0 & \text{in} & \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right), \\ u^n_t(0) &= u^n_1 &= \sum_{i=1}^n (a^n_i)_t(0) w_i &\to u_1 & \text{in} & H^1_{\Gamma_0}(\Omega), \\ v^n(0) &= v^n_0 &= \sum_{i=1}^n b^n_i(0) \widetilde{w}_i &\to v_0 & \text{in} & (H^2(\Gamma_1) \cap H^1(\Gamma_1)), \\ v^n_t(0) &= v^n_1 &= \sum_{i=1}^n (b^n_i)_t(0) \widetilde{w}_i &\to v_1 & \text{in} & H^1(\Gamma_1), \\ z^n_1(\rho,0) &= z^n_{0_1} &= \sum_{i=1}^n c^n_i(0) \phi_i &\to f_{0_1} & \text{in} & H^1_{\Gamma_0}(\Omega; H^1(0,1)), \\ z^n_2(\rho,0) &= z^n_{0_2} &= \sum_{i=1}^n d^n_i(0) \widetilde{\phi}_i &\to f_{0_2} & \text{in} & H^1\big(\Gamma_1; H^1(0,1)\big). \end{cases} \tag{15}$$

The local existence of solutions of the problem (12)–(15) is standard by the theory of ordinary differential equations, we can conclude that there is a $t_n > 0$ such that in $[0, t_n]$, the problem (12)–(15) has a unique local solution which can be extended to a maximal interval $[0, T]$ (with $0 < T \le \infty$) by Zorn's lemma, since the nonlinear terms in (12) are locally Lipschitz continuous.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (u^n, v^n, z^n_1, z^n_2).

The first estimate

Since the sequences $(u^n_0)_n$, $(u^n_1)_n$, $(v^n_0)_n$, $(v^n_1)_n$, $(z^n_{0_1})_n$ and $(z^n_{0_2})_n$ converge, the standard calculations, using (12)–(15) similar to those used to find (10), yield a number M_1 independant of n such that

$$\begin{aligned} & E_n(t) + a_1 \int_0^t \int_\Omega u^n_t g_1(u^n_t) dx ds + a_3 \int_0^t \int_\Omega z^n_1(x,1,s) g_1(z^n_1(x,1,s)) dx ds \\ & + a_2 \int_0^t \int_{\Gamma_1} v^n_t g_2(v^n_t) d\sigma ds + a_4 \int_0^t \int_{\Gamma_1} z^n_2(x,1,s) g_2(z^n_2(x,1,s)) d\sigma ds \\ \le \; & E_n(0) \le M_1, \end{aligned} \tag{16}$$

where

$$\begin{aligned} E_n(t) \;=\; & \frac{1}{2}\Big(\|u^n_t\|^2 + \|\nabla u^n\|^2 + \|v^n_t\|^2_{\Gamma_1} + \|\nabla_T v^n\|^2_{\Gamma_1}\Big) \\ & + \xi \int_\Omega \Big(\int_0^1 G_1(z^n_1(x,\rho,t) d\rho\Big) dx + \zeta \int_{\Gamma_1} \Big(\int_0^1 G_2(z^n_2(x,\rho,t) d\rho\Big) d\sigma \end{aligned}$$

and $a_i, i = 1, ..., 4$ are defined in Lemmma 1.

The estimate (16) imply that the solution $(u^n, v^n, z^n_1, z^n_2)_n$ exists globally in $[0, +\infty)$.

Estimate (16) yields for any $T > 0$

$$u^n \text{ is bounded in } L^\infty\Big(0, T; H^1_{\Gamma_0}(\Omega)\Big), \tag{17}$$

$$v^n \text{ is bounded in } L^\infty\Big(0, T; H^1(\Gamma_1)\Big), \tag{18}$$

$$u^n_t \text{ is bounded in } L^\infty\Big(0, T; L^2(\Omega)\Big), \tag{19}$$

$$v^n_t \text{ is bounded in } L^\infty\Big(0, T; L^2(\Gamma_1)\Big), \tag{20}$$

$$u_t^n g_1(u_t^n) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{21}$$

$$v_t^n g_2(v_t^n) \text{ is bounded in } L^1(\Gamma_1\times(0,T)), \tag{22}$$

$$G_1(z_1^n)) \text{ is bounded in } L^\infty\Big(0,T;L^1(\Omega\times(0,1))\Big), \tag{23}$$

$$G_2(z_2^n)) \text{ is bounded in } L^\infty\Big(0,T;L^1(\Gamma_1\times(0,1))\Big), \tag{24}$$

$$z_1^n(x,1,t)g_1(z_1^n(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{25}$$

$$z_2^n(x,1,t)g_2(z_2^n(x,1,t)) \text{ is bounded in } L^1(\Gamma_1\times(0,T)). \tag{26}$$

The second estimate

We need to estimate $u_{tt}^n(0)$ and $v_{tt}^n(0)$ in norms $L^2(\Omega)$ and $L^2(\Gamma_1)$ respectively. By taking $t=0$ and considering $w_i = u_{tt}^n(0)$ and $\widetilde{w}_i = v_{tt}^n(0)$ in (12), we get

$$\begin{aligned} &\|u_{tt}^n(0)\|^2 + (\nabla u_0^n, \nabla u_{tt}^n(0)) + \mu_1(g_1(u_1^n), u_{tt}^n(0)) + \mu_2(g_1(z_{0_1}^n), u_{tt}^n(0)) \\ &+\|v_{tt}^n(0)\|_{\Gamma_1}^2 + (\nabla_T v_0^n, \nabla_T v_{tt}^n(0))_{\Gamma_1} + \mu_1'(g_2(v_1^n), v_{tt}^n(0))_{\Gamma_1} + \mu_2'(g_2(z_{0_2}^n), v_{tt}^n(0))_{\Gamma_1} \\ =\ & 0. \end{aligned} \tag{27}$$

We have the equalities

$$(\nabla u_0^n, \nabla u_{tt}^n(0)) = -(\Delta u_0^n, u_{tt}^n(0)) + (\partial_\nu u_0^n, v_{tt}^n(0))_{\Gamma_1}, \tag{28}$$

$$(\nabla_T v_0^n, \nabla_T v_{tt}^n(0))_{\Gamma_1} = -(\Delta_T v_0^n, v_{tt}^n(0))_{\Gamma_1}. \tag{29}$$

Employing Young's inequality on (28) and (29) and using the fact that if $u_0^n \in \Big(H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)\Big)$, then $\partial_\nu u_0^n \in H^{1/2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$, hence $\partial_\nu u_0^n \in L^2(\Gamma_1)$, thus

$$\begin{aligned} (\nabla u_0^n, \nabla u_{tt}^n(0)) &= -(\Delta u_0^n, u_{tt}^n(0)) + (\partial_\nu u_0^n, v_{tt}^n(0))_{\Gamma_1} \\ &\le \frac{1}{4\varepsilon}\|\Delta u_0^n\|^2 + \frac{1}{4\varepsilon}\|\partial_\nu u_0^n\|_{\Gamma_1}^2 + \varepsilon\|u_{tt}^n(0)\|^2 + \varepsilon\|v_{tt}^n(0\|_{\Gamma_1}^2, \end{aligned} \tag{30}$$

$$-(\Delta_T v_0^n, v_{tt}^n(0))_{\Gamma_1} \le \frac{1}{4\varepsilon}\|\Delta_T v_0^n\|_{\Gamma_1}^2 + \varepsilon\|v_{tt}^n(0)\|_{\Gamma_1}^2, \tag{31}$$

$$\mu_1(g_1(u_1^n), u_{tt}^n(0)) \le \frac{\mu_1^2}{4\varepsilon}\|g_1(u_1^n)\|^2 + \varepsilon\|u_{tt}^n(0)\|^2, \tag{32}$$

$$\mu_1'(g_2(v_1^n), v_{tt}^n(0))_{\Gamma_1} \le \frac{(\mu_1')^2}{4\varepsilon}\|g_2(v_1^n)\|_{\Gamma_1}^2 + \varepsilon\|v_{tt}^n(0)\|_{\Gamma_1}^2, \tag{33}$$

$$\mu_2(g_1(z_{0_1}^n), u_{tt}^n(0)) \le \frac{\mu_2^2}{4\varepsilon}\Big\|g_1(z_{0_1}^n)\Big\|^2 + \varepsilon\|u_{tt}^n(0)\|^2, \tag{34}$$

$$\mu_2'(g_2(z_{0_2}^n), v_{tt}^n(0))_{\Gamma_1} \le \frac{(\mu_2')^2}{4\varepsilon}\Big\|g_2(z_{0_2}^n)\Big\|_{\Gamma_1}^2 + \varepsilon\|v_{tt}^n(0)\|_{\Gamma_1}^2, \tag{35}$$

by reinjecting (30)–(35) in (27), with a suitable choice of ε and since $(g_1(u_1^n))_n$, $(g_1(z_{0_1}^n))_n$ and $(g_2(v_1^n))_n$, $(g_2(z_{0_2}^n))_n$ are bounded in $L^2(\Omega)$ and $L^2(\Gamma_1)$ respectively by (**A1**), (**A2**) and initial data (15), we get

$$\|u_{tt}^n(0)\| + \|v_{tt}^n(0)\|_{\Gamma_1} \leq M_2, \tag{36}$$

where M_2 is a positive constant independent of n and depends on the initial datas.

Next, differentiating (12) with respect to t, multiplying the resulting equation by $(a_i^n)_{tt}(t)$ in Ω and by $(b_i^n)_{tt}(t)$ on Γ_1 and summing over i from 1 to n, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left\{\|u_{tt}^n\|^2 + \|\nabla u_t^n\|^2 + \|v_{tt}^n\|_{\Gamma_1}^2 + \|\nabla_T v_t^n\|_{\Gamma_1}^2\right\} \\ & +\mu_1 \int_\Omega |u_{tt}^n|^2 (g_1)_t(u_t^n)dx + \mu_2 \int_\Omega u_{tt}^n (z_1^n)_t(x,1,t)(g_1)_t(z_1^n(x,1,t))dx \\ & +\mu_1' \int_{\Gamma_1} |v_{tt}^n|^2 (g_2)_t(v_t^n)d\sigma + \mu_2' \int_{\Gamma_1} v_{tt}^n (z_2^n)_t(x,1,t)(g_2)_t(z_2^n(x,1,t))d\sigma \\ = \ & 0. \end{aligned} \tag{37}$$

Differentiating (13) with respect to t, multiplying the resulting equation by $(c_i^n)_t(t)$ and summing over i from 1 to n, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\left\|z_{1_t}^n(\rho,t)\right\|_{L^2(\Omega\times(0,1))}^2 + \frac{1}{2}\frac{d}{d\rho}\left\|z_{1_t}^n(\rho,t)\right\|_{L^2(\Omega\times(0,1))}^2 = 0. \tag{38}$$

Analogously, differentiating (14) with respect to t, multiplying the resulting equation by $(d_i^n)_t(t)$ and summing over i from 1 to n, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|z_{2_t}^n(\rho,t)\|_{L^2(\Gamma_1\times(0,1))}^2 + \frac{1}{2}\frac{d}{d\rho}\|z_{2_t}^n(\rho,t)\|_{L^2(\Gamma_1\times(0,1))}^2 = 0. \tag{39}$$

Taking the sum of (37)–(39), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\Big\{\|u_{tt}^n\|^2 + \|\nabla u_t^n\|^2 + \|v_{tt}^n\|_{\Gamma_1}^2 + \|\nabla_T v_t^n\|_{\Gamma_1}^2 \\ & +\tau\|(z_1^n)_t(\rho,t)\|_{L^2(\Omega\times(0,1))}^2 + \tau\|(z_2^n)_t(\rho,t)\|_{L^2(\Gamma_1\times(0,1))}^2\Big\} \\ & +\mu_1 \int_\Omega |u_{tt}^n|^2 (g_1)_t(u_t^n)dx + \frac{1}{2}\int_\Omega |(z_1^n)_t(x,1,t)|^2 dx \\ & +\mu_1' \int_{\Gamma_1} |v_{tt}^n|^2 (g_2)_t(v_t^n)d\sigma + \frac{1}{2}\int_{\Gamma_1} |(z_2^n)_t(x,1,t)|^2 d\sigma \\ = \ & \frac{1}{2}\|u_{tt}^n\|^2 - \mu_2 \int_\Omega u_{tt}^n (z_1^n)_t(x,1,t)(g_1)_t(z_1^n(x,1,t))dx \\ & +\frac{1}{2}\|v_{tt}^n\|_{\Gamma_1}^2 - \mu_2' \int_{\Gamma_1} v_{tt}^n (z_2^n)_t(x,1,t)(g_2)_t(z_2^n(x,1,t))d\sigma. \end{aligned} \tag{40}$$

Using (4) and Young's inequality, we obtain

$$\begin{aligned} & \mu_2 \int_\Omega u_{tt}^n (z_1^n)_t(x,1,t)(g_1)_t(z_1^n(x,1,t))dx \\ \leq \ & \varepsilon \int_\Omega |(z_1^n)_t(x,1,t)|^2 dx + \frac{(\mu_2 c)^2}{4\varepsilon}\|u_{tt}^n\|^2, \end{aligned} \tag{41}$$

$$\begin{aligned} & \mu_2' \int_{\Gamma_1} v_{tt}^n (z_2^n)_t(x,1,t)(g_2)_t(z_2^n(x,1,t))d\sigma \\ \leq \ & \varepsilon \int_{\Gamma_1} |(z_2^n)_t(x,1,t)|^2 d\sigma + \frac{(\mu_2' c)^2}{4\varepsilon}\|v_{tt}^n\|_{\Gamma_1}^2. \end{aligned} \tag{42}$$

Reinjecting (41) and (42) in (40) and choosing ε small enough, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\|u_{tt}^n\|^2+\|\nabla u_t^n\|^2+\|v_{tt}^n\|_{\Gamma_1}^2+\|\nabla_T v_t^n\|_{\Gamma_1}^2\\
&+\tau\|(z_1^n)_t(\rho,t)\|_{L^2(\Omega\times(0,1))}^2+\tau\|(z_2^n)_t(\rho,t)\|_{L^2(\Gamma_1\times(0,1))}^2\Big\}\\
&+\mu_1\int_\Omega|u_{tt}^n|^2(g_1)_t(u_t^n)dx+c\int_\Omega|(z_1^n)_t(x,1,t)|^2dx\\
&+\mu_1'\int_{\Gamma_1}|v_{tt}^n|^2(g_2)_t(v_t^n)d\sigma+c\int_{\Gamma_1}|(z_2^n)_t(x,1,t)|^2d\sigma\\
\le\;& c'\Big\{\|u_{tt}^n\|^2+\|v_{tt}^n\|_{\Gamma_1}^2\Big\}.
\end{aligned}$$

Integrating the last inequality over $(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big\{\|u_{tt}^n\|^2+\|\nabla u_t^n\|^2+\|v_{tt}^n\|_{\Gamma_1}^2+\|\nabla_T v_t^n\|_{\Gamma_1}^2\\
&+\tau\|(z_1^n)_t(\rho,t)\|_{L^2(\Omega\times(0,1))}^2+\tau\|(z_2^n)_t(\rho,t)\|_{L^2(\Gamma_1\times(0,1))}^2\Big\}\\
&+\mu_1\int_0^t\int_\Omega|u_{tt}^n|^2(g_1)_t(u_t^n)dxds+c\int_0^t\int_\Omega(z_1^n)_t|(x,1,t)|^2dxds\\
&+\mu_1'\int_0^t\int_{\Gamma_1}|v_{tt}^n|^2(g_2)_t(v_t^n)d\sigma ds+c\int_0^t\int_{\Gamma_1}|(z_2^n)_t(x,1,t)|^2d\sigma ds\\
\le\;&\frac{1}{2}\Big\{\|u_{tt}^n(0)\|^2+\|v_{tt}^n(0)\|_{\Gamma_1}^2+\|\nabla u_1^n\|^2+\|\nabla_T v_1^n\|_{\Gamma_1}^2\\
&+\tau\|(z_1^n)_t(x,\rho,0)\|_{L^2(\Omega\times(0,1))}^2+\tau\|(z_2^n)_t(x,\rho,0)\|_{L^2(\Gamma_1\times(0,1))}^2\Big\}\\
&+c'\Big\{\int_0^t\|u_{tt}^n(s)\|_\Omega^2ds+\int_0^t\|v_{tt}^n(s)\|_{\Gamma_1}^2ds\Big\}.
\end{aligned}$$

Using (15) and (36), then we use Gronwall's lemma to get

$$\begin{aligned}
&\|u_{tt}^n\|^2+\|\nabla u_t^n\|^2+\|v_{tt}^n\|_{\Gamma_1}^2+\|\nabla_T v_t^n\|_{\Gamma_1}^2+\tau\|(z_1^n)_t(\rho,t)\|_{L^2(\Omega\times(0,1))}^2\\
&+\tau\|(z_2^n)_t(\rho,t)\|_{L^2(\Gamma_1\times(0,1))}^2+\mu_1\int_0^t\int_\Omega|u_{tt}^n|^2(g_1)_t(u_t^n)dxds\\
&+c\int_0^t\int_\Omega(z_1^n)_t|(x,1,s)|^2dxds+\mu_1'\int_0^t\int_{\Gamma_1}|v_{tt}^n|^2(g_2)_t(v_t^n)d\sigma ds\\
&+c\int_0^t\int_{\Gamma_1}|(z_2^n)_t(x,1,s)|^2d\sigma ds\\
\le\;& M_3,
\end{aligned}$$

where M_3 is independent of n and for all $t\in[0,T]$. Therefore, we conclude that

$$u_t^n \text{ is bounded in } L^\infty\Big(0,T;H^1_{\Gamma_0}(\Omega)\Big), \tag{43}$$

$$v_t^n \text{ is bounded in } L^\infty\Big(0,T;H^1(\Gamma_1)\Big), \tag{44}$$

$$u_{tt}^n \text{ is bounded in } L^\infty\Big(0,T;L^2(\Omega)\Big), \tag{45}$$

$$v_{tt}^n \text{ is bounded in } L^\infty\Big(0,T;L^2(\Gamma_1)\Big), \tag{46}$$

$$z_{1_t}^n \text{ is bounded in } L^\infty(0,T;L^2(\Omega\times(0,1))), \tag{47}$$

$$z^n_{2_t} \text{ is bounded in } L^\infty(0,T;L^2(\Gamma_1\times(0,1)). \tag{48}$$

Estimate for $(z^n_1)_n$ and $(z^n_2)_n$

Replacing ϕ_i by $-\Delta\phi_i$ in (13), multiplying the resulting equation by $c^n_i(t)$ and summing over i from 1 to n, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla z^n_1(\rho,t)\|^2_{L^2(\Omega\times(0,1))}+\frac{1}{2}\frac{d}{d\rho}\|\nabla z^n_1(\rho,t)\|^2_{L^2(\Omega\times(0,1))}=0. \tag{49}$$

Similarly, replacing $\widetilde{\phi}_i$ by $-\Delta_T\widetilde{\phi}_i$ in (14), multiplying the resulting equation by $d^n_i(t)$ and summing over i from 1 to n, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla_T z^n_2(\rho,t)\|^2_{L^2(\Gamma_1\times(0,1))}+\frac{1}{2}\frac{d}{d\rho}\|\nabla_T z^n_2(\rho,t)\|^2_{L^2(\Gamma_1\times(0,1))}=0. \tag{50}$$

Combining (49) and (50), we have

$$\begin{aligned}&\frac{\tau}{2}\frac{d}{dt}\|\nabla z^n_1(\rho,t)\|^2_{L^2(\Omega\times(0,1))}+\frac{\tau}{2}\frac{d}{dt}\|\nabla_T z^n_2(\rho,t)\|^2_{L^2(\Gamma_1\times(0,1))}\\&+\frac{1}{2}\left\{\int_\Omega|\nabla z^n_1(x,1,t)|^2dx+\int_{\Gamma_1}|\nabla_T z^n_2(x,1,t)|^2d\sigma\right\}\\=\;&\frac{1}{2}\left\{\|\nabla u^n\|^2+\|\nabla_T v^n\|^2_{\Gamma_1}\right\}.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we have

$$\begin{aligned}&\frac{\tau}{2}\|\nabla z^n_1(\rho,t)\|^2_{L^2(\Omega\times(0,1))}+\frac{\tau}{2}\|\nabla_T z^n_2(\rho,t)\|^2_{L^2(\Gamma_1\times(0,1))}\\\le\;&e^{cT}\left\{\frac{\tau}{2}\|\nabla z^n_1(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))}+\frac{\tau}{2}\|\nabla_T z^n_2(x,\rho,0)\|^2_{L^2(\Gamma_1\times(0,1))}\right\},\end{aligned}$$

for all $t\in[0,T]$. Therefore, we conclude that

$$z^n_1 \text{ is bounded in } L^\infty\left(0,T;H^1_{\Gamma_0}\left(\Omega;L^2(0,1)\right)\right), \tag{51}$$

$$z^n_2 \text{ is bounded in } L^\infty\left(0,T;H^1\left(\Gamma_1;L^2(0,1)\right)\right). \tag{52}$$

The passing to the limit

Applying Dunford-Petti's theorem, we conclude that there exists subsequences of $(u^n)_n$, $(v^n)_n$, $(z^n_1)_n$ and $(z^n_2)_n$ which we still denote by $(u^n)_n$, $(v^n)_n$, $(z^n_1)_n$ and $(z^n_2)_n$ respectively, such that from (17), (43) and (45), we get

$$(u^n,u^n_t,u^n_{tt})\rightharpoonup(u,u_t,u_{tt}) \text{ weakly star in } L^\infty\left(0,T;\left[H^1_{\Gamma_0}(\Omega)\right]^2\times L^2(\Omega)\right), \tag{53}$$

from (18), (44) and (46), we obtain

$$(v^n,v^n_t,v^n_{tt})\rightharpoonup(v,v_t,v_{tt}) \text{ weakly star in } L^\infty\left(0,T;\left[H^1(\Gamma_1)\right]^2\times L^2(\Gamma_1)\right), \tag{54}$$

from (51) and (52), we find

$$z^n_1\rightharpoonup z_1 \text{ weakly star in } L^\infty\left(0,T;H^1_{\Gamma_0}\left(\Omega;L^2(0,1)\right)\right), \tag{55}$$

$$z^n_2\rightharpoonup z_2 \text{ weakly star in } L^\infty\left(0,T;H^1\left(\Gamma_1;L^2(0,1)\right)\right), \tag{56}$$

from (47) and (48), we get

$$z^n_{1_t} \rightharpoonup z_{1_t} \text{ weakly star in } L^\infty(0,T;L^2(\Omega\times(0,1))), \tag{57}$$

$$z^n_{2_t} \rightharpoonup z^n_{2_t} \text{ weakly star in } L^\infty(0,T;L^2(\Gamma_1\times(0,1))), \tag{58}$$

and from (19)–(26), we have

$$g_1(u^n_t) \rightharpoonup \chi_1 \text{ weakly star in } L^2((0,T)\times\Omega),$$
$$g_2(v^n_t) \rightharpoonup \chi_2 \text{ weakly star in } L^2((0,T)\times\Gamma_1),$$
$$g_1(z^n_1(x,1,t)) \rightharpoonup \Psi_1 \text{ weakly star in } L^2((0,T)\times\Omega),$$
$$g_2(z^n_2(x,1,t)) \rightharpoonup \Psi_2 \text{ weakly star in } L^2((0,T)\times\Gamma_1).$$

Thanks to Aubin-Lions's theorem, (see [5]), we deduce that there exists subsequences which we still denote $(u^n)_n$, $(v^n)_n$, $(z^n_1)_n$ and $(z^n_2)_n$, such that

$$u^n \to u \text{ strongly in } L^2\Big(0,T;L^2(\Omega)\Big), \tag{59}$$

$$u^n_t \to u_t \text{ strongly in } L^2\Big(0,T;L^2(\Omega)\Big), \tag{60}$$

$$v^n \to v \text{ strongly in } L^2\Big(0,T;L^2(\Gamma_1)\Big), \tag{61}$$

$$v^n_t \to v_t \text{ strongly in } L^2\Big(0,T;L^2(\Gamma_1)\Big), \tag{62}$$

$$z^n_1 \to z_1 \text{ strongly in } L^2(\Omega\times(0,1)\times(0,T)), \tag{63}$$

$$z^n_2 \to z_2 \text{ strongly in } L^2(\Gamma_1\times(0,1)\times(0,T)). \tag{64}$$

Analysis of the nonlinear terms

Denote by $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma_1\times(0,T)$.

We can deduce from (60) and (62)

$$u^n_t \to u_t \text{ almost everywhere on } Q, \tag{65}$$

$$v^n_t \to v_t \text{ almost everywhere on } \Sigma, \tag{66}$$

and

$$\begin{aligned} z^n_1 &\to z_1 \text{ strongly in } L^2\Big(0,T;L^2(\Omega)\Big) \text{ and } a.e \text{ on } Q,\\ z^n_2 &\to z_2 \text{ strongly in } L^2\Big(0,T;L^2(\Gamma_1)\Big) \text{ and } a.e \text{ on } \Sigma. \end{aligned}$$

We have the following two lemmas, (for the proof, see [4]):

Lemma 2. *For each $T>0$, $g_1(u_t), g_1(z_1(.,1,.))\in L^1(Q)$ and $g_2(v_t), g_2(z_2(.,1,.))\in L^1(\Sigma)$, we have*

$$\|g_1(u_t)\|_{L^1(Q)}, \|g_1(z_1(.,1,.))\|_{L^1(Q)} \le A_1$$

and

$$\|g_2(v_t)\|_{L^1(\Sigma)}, \|g_2(z_2(.,1,.))\|_{L^1(\Sigma)} \le A_2,$$

where A_1 and A_2 are constants independent of t.

Lemma 3. *We have the following convergences*

$$\begin{cases} g_1(u_t^n) \to g_1(u_t) \text{ in } L^1(\Omega \times (0,T)), \\ g_2(v_t^n) \to g_2(v_t) \text{ in } L^1(\Gamma_1 \times (0,T)). \end{cases}$$

$$\begin{cases} g_1(z_1^n) \to g_1(z_1) \text{ in } L^1(\Omega \times (0,T)), \\ g_2(z_2^n) \to g_2(z_2) \text{ in } L^1(\Gamma_1 \times (0,T)). \end{cases}$$

Hence from Lemma 3, we deduce that

$$\begin{cases} g_1(u_t^n) \rightharpoonup \chi_1 = g_1(u_t) \text{ weakly in } L^2(\Omega \times (0,T)), \\ g_2(v_t^n) \rightharpoonup \chi_2 = g_2(v_t) \text{ weakly in } L^2(\Gamma_1 \times (0,T)), \end{cases} \tag{67}$$

and

$$\begin{cases} g_1(z_1^n(x,1,t)) \rightharpoonup \Psi_1 = g_1(z_1(x,1,t)) \text{ weakly in } L^2(\Omega \times (0,T)), \\ g_2(z_2^n(x,1,t)) \rightharpoonup \Psi_2 = g_2(z_2(x,1,t)) \text{ weakly in } L^2(\Gamma_1 \times (0,T)). \end{cases} \tag{68}$$

Now, returning to (12) and using standard arguments, we can show from the above estimates that

$$u_{tt} - \triangle u + \mu_1 g_1(u_t) + \mu_2 g_1(z_1(.,1,.)) = 0, \text{ in } \mathcal{D}'(\Omega \times (0,T)). \tag{69}$$

Since u_{tt}, $g_1(u_t)$ and $g_1(z_1(.,1,.)) \in L^2(0,T;L^2(\Omega))$, we obtain from identity (69)

$$\triangle u \in L^2\Big(0,T;L^2(\Omega)\Big),$$

and therefore identity (69) yields

$$u_{tt} - \triangle u + \mu_1 g_1(u_t) + \mu_2 g_1(z_1(.,1,.)) = 0, \text{ in } L^2\Big(0,T;L^2(\Omega)\Big). \tag{70}$$

Taking (70) into account and making use of the generalized Green's formula, we deduce that

$$\partial_\nu u - \Delta_T v = -\mu_1' g_2(v_t) - \mu_2' g_2(z_2(.,1,.)) - v_{tt}, \text{ in } \mathcal{D}'\Big(0,T;H^{-\frac{1}{2}}(\Gamma_1)\Big),$$

and since v_{tt}, $g_2(v_t)$ and $g_2(z_2(.,1,.)) \in L^2(0,T;L^2(\Gamma_1))$, we infer that

$$\partial_\nu u - \Delta_T v = -\mu_1' g_2(v_t) - \mu_2' g_2(z_2(.,1,.)) - v_{tt}, \text{ in } L^2\Big(0,T;L^2(\Gamma_1)\Big),$$

then

$$v_{tt} + \partial_\nu u - \Delta_T v + \mu_1' g_2(v_t) + \mu_2' g_2(z_2(.,1,.)) = 0, \text{ in } L^2\Big(0,T;L^2(\Gamma_1)\Big).$$

Exploiting the convergences (55)–(58), (63) and (64), we pass to the limit in (13) and (14) to obtain

$$\int_0^T \int_0^1 \int_\Omega (\tau z_{1_t} + z_{1\rho})\vartheta_1 dx d\rho dt = 0, \qquad \forall \vartheta_1 \in L^2\Big(0,T;H^1_{\Gamma_0}(\Omega \times (0,1))\Big),$$

$$\int_0^T \int_0^1 \int_{\Gamma_1} (\tau z_{2_t} + z_{2\rho})\vartheta_2 d\sigma d\rho dt = 0, \qquad \forall \vartheta_2 \in L^2\Big(0,T;H^1(\Gamma_1 \times (0,1))\Big).$$

Uniqueness.

Let (u, v, z_1, z_2) and $(\widetilde{u}, \widetilde{v}, \widetilde{z}_1, \widetilde{z}_2)$ be two solutions of problem (6). Then $(U, V, Z_1, Z_2) = (u, v, z_1, z_2) - (\widetilde{u}, \widetilde{v}, \widetilde{z}_1, \widetilde{z}_2)$ verifies the following system of equations:

$$\begin{cases} U_{tt} - \Delta U + \mu_1 g_1(u_t) - \mu_1 g_1(\widetilde{u}_t) & & \\ \quad + \mu_2 g_1(z_1(x,1,t)) - \mu_2 g_1(\widetilde{z}_1(x,1,t)) = 0, & \text{in} & \Omega \times (0,\infty), \\ V_{tt} + \partial_\nu U - \Delta_T V + \mu_1' g_2(v_t) - \mu_1' g_2(\widetilde{v}_t) & & \\ \quad + \mu_2' g_2(z_2(x,1,t)) - \mu_2' g_2(\widetilde{z}_2(x,1,t)) = 0, & \text{on} & \Gamma_1 \times (0,\infty), \\ \tau Z_{1_t}(x,\rho,t) + Z_{1_\rho}(x,\rho,t) = 0, & \text{in} & \Omega \times (0,1) \times (0,\infty), \\ \tau Z_{2_t}(x,\rho,t) + Z_{2_\rho}(x,\rho,t) = 0, & \text{on} & \Gamma_1 \times (0,1) \times (0,\infty), \\ U = V, & \text{on} & \Gamma \times (0,\infty), \\ U = 0, & \text{on} & \Gamma_0 \times (0,\infty), \\ Z_1(x,0,t) = U_t(x,t), & \text{in} & \Omega \times (0,\infty), \\ Z_2(x,0,t) = V_t(x,t), & \text{on} & \Gamma_1 \times (0,\infty), \\ (U(0), V(0)) = (0,0), & \text{in} & \Omega \times \Gamma, \\ (U_t(0), V_t(0)) = (0,0), & \text{in} & \Omega \times \Gamma, \\ Z_1(x,\rho,0) = 0, & \text{in} & \Omega \times (0,1), \\ Z_2(x,\rho,0) = 0, & \text{on} & \Gamma_1 \times (0,1). \end{cases} \tag{71}$$

Multiplying the first equation of (71) by U_t, we have

$$\begin{aligned} & (U_{tt}, U_t) - (\Delta U, U_t) + \mu_1(g_1(u_t) - g_1(\widetilde{u}_t), U_t) \\ & + \mu_2(g_1(z_1(x,1,t)) - g_1(\widetilde{z}_1(x,1,t)), U_t) \\ = \ & 0, \end{aligned} \tag{72}$$

next, integrating over Ω, we get

$$-(\Delta U, U_t) = (\nabla U, \nabla U_t) - (\partial_\nu U, V_t)_{\Gamma_1},$$

then, (72) becomes

$$\begin{aligned} & (U_{tt}, U_t) + (\nabla U, \nabla U_t) - (\partial_\nu U, V_t)_{\Gamma_1} + \mu_1(g_1(u_t) - g_1(\widetilde{u}_t), U_t) \\ & + \mu_2(g_1(z_1(x,1,t)) - g_1(\widetilde{z}_1(x,1,t)), U_t) \\ = \ & 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|U_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 - (\partial_\nu U, V_t)_{\Gamma_1} + \mu_1(g_1(u_t) - g_1(\widetilde{u}_t), U_t) \\ & + \mu_2(g_1(z_1(x,1,t)) - g_1(\widetilde{z}_1(x,1,t)), U_t) \\ = \ & 0. \end{aligned} \tag{73}$$

Multiplying the second equation of (71) by V_t, we have

$$\begin{aligned} & (V_{tt}, V_t)_{\Gamma_1} + (\partial_\nu U, V_t)_{\Gamma_1} - (\Delta_T V, V_t)_{\Gamma_1} + \mu_1'(g_2(v_t) - g_2(\widetilde{v}_t), V_t)_{\Gamma_1} \\ & + \mu_2'(g_2(z_2(x,1,t)) - g_2(\widetilde{z}_2(x,1,t)), V_t)_{\Gamma_1} \\ = \ & 0, \end{aligned} \tag{74}$$

next, integrating over Γ_1, we get

$$-(\Delta_T V, V_t)_{\Gamma_1} = (\nabla_T V, \nabla_T V_t)_{\Gamma_1},$$

then, (74) becomes

$$\begin{aligned}
&(V_{tt}, V_t)_{\Gamma_1} + (\partial_\nu U, V_t)_{\Gamma_1} + (\nabla_T V, \nabla_T V_t)_{\Gamma_1} + \mu_1'(g_2(v_t) - g_2(\widetilde{v}_t), V_t)_{\Gamma_1} \\
&+\mu_2'(g_2(z_2(x,1,t)) - g_2(\widetilde{z}_2(x,1,t)), V_t)_{\Gamma_1} \\
=\;& 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V_t\|_{\Gamma_1}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla_T V\|_{\Gamma_1}^2 + (\partial_\nu U, V_t)_{\Gamma_1} + \mu_1'(g_2(v_t) - g_2(\widetilde{v}_t), V_t)_{\Gamma_1} \\
&+\mu_2'(g_2(z_2(x,1,t)) - g_2(\widetilde{z}_2(x,1,t)), V_t)_{\Gamma_1} \\
=\;& 0.
\end{aligned} \tag{75}$$

Now, summing (73) and (75), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 + \frac{1}{2}\frac{d}{dt}\|V_t\|_{\Gamma_1}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla_T V\|_{\Gamma_1}^2 \\
&+\mu_1(g_1(u_t) - g_1(\widetilde{u}_t), U_t) + \mu_1'(g_2(v_t) - g_2(\widetilde{v}_t), V_t)_{\Gamma_1} \\
&+\mu_2(g_1(z_1(x,1,t)) - g_1(\widetilde{z}_1(x,1,t)), U_t) \\
&+\mu_2'(g_2(z_2(x,1,t)) - g_2(\widetilde{z}_2(x,1,t)), V_t)_{\Gamma_1} \\
=\;& 0.
\end{aligned} \tag{76}$$

Similarly, multiplying the third and fourth equation of (71) by respectively Z_1 and Z_2, integrating over $\Omega \times (0,1)$ and $\Gamma_1 \times (0,1)$, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|Z_1(x,\rho,t)\|^2 d\rho + \frac{1}{2}\Big\{\|Z_1(x,1,t)\|^2 - \|U_t(x,t)\|^2\Big\} = 0, \tag{77}$$

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|Z_2(x,\rho,t)\|_{\Gamma_1}^2 d\rho + \frac{1}{2}\Big\{\|Z_2(x,1,t)\|_{\Gamma_1}^2 - \|V_t(x,t)\|_{\Gamma_1}^2\Big\} = 0. \tag{78}$$

From (76)–(78), and using Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\|U_t\|^2 + \|\nabla U\|^2 + \|V_t\|_{\Gamma_1}^2 + \|\nabla_T V\|_{\Gamma_1}^2 \\
&+\tau\int_0^1 \|Z_1(x,\rho,t)\|^2 d\rho + \tau\int_0^1 \|Z_2(x,\rho,t)\|_{\Gamma_1}^2 d\rho\Big\} \\
&+\mu_1(g_1(u_t) - g_1(\widetilde{u}_t), U_t) + \mu_1'(g_2(v_t) - g_2(\widetilde{v}_t), V_t)_{\Gamma_1} \\
=\;& -\mu_2(g_1(z_1(x,1,t)) - g_1(\widetilde{z}_1(x,1,t)), U_t) \\
&-\mu_2'(g_2(z_2(x,1,t)) - g_2(\widetilde{z}_2(x,1,t)), V_t)_{\Gamma_1} \\
&+\frac{1}{2}\|U_t(x,t)\|^2 + \frac{1}{2}\|V_t(x,t)\|_{\Gamma_1}^2 \\
\leq\;& \frac{1}{2}\|U_t(x,t)\|^2 + \|g_1(z_1(x,1,t)) - g_1(\widetilde{z}_1(x,1,t))\|^2 \|U_t(x,t)\|^2 \\
&+\frac{1}{2}\|V_t(x,t)\|_{\Gamma_1}^2 + \|g_2(z_2(x,1,t)) - g_2(\widetilde{z}_2(x,1,t))\|_{\Gamma_1}^2 \|V_t(x,t)\|_{\Gamma_1}^2,
\end{aligned}$$

next, using condition (4) and Young's inequality, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\|U_t\|^2 + \|\nabla U\|^2 + \|V_t\|_{\Gamma_1}^2 + \|\nabla_T V\|_{\Gamma_1}^2 \\
&+\tau\int_0^1 \|Z_1(x,\rho,t)\|^2 d\rho + \tau\int_0^1 \|Z_2(x,\rho,t)\|_{\Gamma_1}^2 d\rho\Big\} \\
\leq\;& c\Big\{\|U_t(x,t)\|^2 + \|V_t(x,t)\|_{\Gamma_1}^2 + \|Z_1(x,1,t)\|^2 + \|Z_2(x,1,t)\|_{\Gamma_1}^2\Big\},
\end{aligned}$$

where $c > 0$, then integrating over $(0, t)$ and using Gronwall's lemma, we conclude that

$$\begin{aligned} & \|U_t\|^2 + \|\nabla U\|^2 + \|V_t\|^2_{\Gamma_1} + \|\nabla_T V\|^2_{\Gamma_1} \\ & + \tau \int_0^1 \|Z_1(x, \rho, t)\|^2 d\rho + \tau \int_0^1 \|Z_2(x, \rho, t)\|^2_{\Gamma_1} d\rho \\ = \; & 0, \end{aligned}$$

which implies $(U, V, Z_1, Z_2) = 0$.

This finishes the proof of Theorem 1. □

3. Conclusions

In this article, we have proved the existence and uniqueness of the solution of a wave equation with dynamic Wentztell type boundary conditions on a part of the boundary Γ_1 of the domain Ω with nonlinear delays in nonlinear dampings in Ω and on Γ_1, using Faedo-Galerkin's method.

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