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Proceeding Paper

Generic Riemannian Maps From Nearly Kaehler Manifold †

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Abstract: In order to generalise semi-invariant Riemannian map, B. Sahin [1] first introduced the idea of "Generic Riemannian maps". We extend the idea of generic Riemannian maps to the case in which the total manifold is nearly Kaehler manifold. We study the integrability conditions for the horizontal distribution although vertical distribution is always integrable. We also study the geometry of foliations of two distributions and obtain the necessary and sufficient condition for generic Riemannian maps to be totally geodesic. Additionaly, we study the generic Riemannian map with umbilical fibers.

Keywords: nearly Kaehler manifold; Riemannian maps; generic Riemannian maps; anti-invariant; semi-invariant Riemannian maps and product manifolds

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1. Introduction

The idea of a Riemannian map between Riemannian manifolds play a key role in differential geometry and this idea was first introduced by Fischer [2] as a generalization of the notions of an isometric immersion and a Riemannian submersion.

Let us consider the smooth map $F : (\mathcal{M}, g_{\mathcal{M}}) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ between Riemannian manifolds $(\mathcal{M}, g_{\mathcal{M}})$ and $(\mathcal{B}, g_{\mathcal{B}})$. Then the tangent bundle of $\mathcal{M}$ has the following decomposition

$$T\mathcal{M} = (kerF_*) \oplus (kerF_*)^{\perp},$$

where the kernal space of F_* is denoted by $kerF_*$ and its orthogonal complement is denoted by $(kerF_*)^{\perp}$. We denote range space of F_* by $rangF_*$ and its orthogonal complement by $(rangF_*)^{\perp}$. Then the tangent bundle $T\mathcal{B}$ of $\mathcal{B}$ has following decomposition

$$T\mathcal{B} = (rangF_*) \oplus (rangF_*)^{\perp}.$$

There are many articles on the geometry of Riemannian map [2,3]. In this paper, we introduce and study generic Riemannian maps from nearly Kaehler manifolds to Riemannian manifolds.

2. Preliminaries

In this section we recall some fundamentals of almost Hermitian manifold, Kaehler manifold, nearly Kaehler manifold and give a brief review of Riemannian maps and generic Riemannian maps.

Let $\mathcal{M}$ be an almost complex manifold with an almost complex structure J and a Riemannian metric $g_{\mathcal{M}}$ satisfying the condition

$$g_{\mathcal{M}}(JX, JY) = g_{\mathcal{M}}(X, Y), \tag{1}$$

for all $X, Y \in \Gamma(T\mathcal{M})$. Then $\mathcal{M}$ is called an almost Hermitian manifold. Let ∇ be the Levi-civita connection on almost Hermitian manifold $\mathcal{M}$, then $\mathcal{M}$ is called a Kaehler manifold if

$$(\nabla_X J)Y = 0, \tag{2}$$

and $\mathcal{M}$ is called a nearly Kaehler manifold if the tensor field ∇J is skew symmetric, i.e.,

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \tag{3}$$

for all $X, Y \in \Gamma(T\mathcal{M})$.

Let $F : (\mathcal{M}, g_{\mathcal{M}}) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a Riemannian map between Riemannian manifolds. Then the geometry of Riemannian maps is characterized by the tensor field T and A, which are B.O'Neils fundamental tensor fields defined for the Riemannian submersion. For arbitrary vector fields E and F,the tensor fields T and$A$ is defined as:

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{4}$$

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F. \tag{5}$$

Using (4) and (5), we have following Lemma

Lemma 1 ([4])**.** *Let $X, Y \in \Gamma(kerF_*)^{\perp}$ and $U, V \in \Gamma(kerF_*)$, we have*

$$\nabla_U V = T_U V + \hat{\nabla}_U V, \tag{6}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + T_U X, \tag{7}$$

$$\nabla_X U = A_X U + \mathcal{V}\nabla_X U, \tag{8}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y, \tag{9}$$

where $\hat{\nabla}_U V = \mathcal{V}\nabla_U V$.

3. Generic Riemannian Maps

In this section we define generic Riemannian maps. We investigate the integrability of the leaves of distribution and obtain the neceassary and sufficient conditions for such maps to be totally geodesic. For such maps, we also obtain decomposition theorem for total manifold.

First, we recall the following definition [1].

Definition 1. *Let us consider a Riemannian map F from an almost Hermitian manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. If the dimension $\mathcal{D}_p$ is constant along $\mathcal{M}$ and it defines a differentiable distribution on $\mathcal{M}$, then we say that F is a generic Riemannian map, where $\mathcal{D}_p$ is the complex subspace of the vertical space where $p \in \mathcal{M}$.*

For a generic Riemannian map,

$$kerF_* = \mathcal{D} \oplus \mathcal{D}^{\perp}, \tag{10}$$

where $\mathcal{D}^{\perp}$ is the orthogonal complementary distribution of $\mathcal{D}$ in $\Gamma(kerF_*)$. For any $U \in \Gamma(kerF_*)$ by the definition of generic Riemannian map, we write

$$JU = \phi U + \omega U, \tag{11}$$

where $\phi U \in (kerF_*)$ and $\omega U \in \Gamma(kerF_*)^\perp$.

We denote the orthogonal complementary distribution of $\omega\mathcal{D}^\perp$ in $\Gamma(kerF_*)^\perp$ by μ. Thus, for any $X \in \Gamma(kerF_*)^\perp$, we write

$$JX = BX + CX, \tag{12}$$

where$BX \in \Gamma(\mathcal{D})^\perp$ and $CX \in \Gamma(\mu)$.

Using (10), for $U \in \Gamma(kerF_*)$, we set

$$JU = P_1U + P_2U + \omega U, \tag{13}$$

where the orthogonal projections from $kerF_*$ to $\mathcal{D}$ and $\mathcal{D}^\perp$ are P_1 and P_2 respectively.

The covariant derivative of a $(1,1)$ tensor field J was firstly defined by S. Ali and T. Fatima [5]. For any arbitrary tangent vector fields E and F on $\mathcal{M}$ we set

$$(\nabla_E J)F = P_EF + Q_EF, \tag{14}$$

where P_EF and Q_EF denote the horizontal and the vertical part of $(\nabla_E J)F$ respectively. If $\mathcal{M}$ is nearly Kaehler manifold then

$$P_EF = -P_FE, \quad Q_EF = -Q_FE. \tag{15}$$

Now we investigate the integrability of distribution.

Theorem 1. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the distribution $\mathcal{D}^\perp$ is integrable if and only if*

$$\hat{\nabla}_V P_2U - \hat{\nabla}_U P_2V + T_V\omega U - T_U\omega V + 2Q_VU \in \Gamma(\mathcal{D})^\perp,$$

for any $U, V \in \Gamma(\mathcal{D})^\perp$.

Proof. For any $U, V \in \Gamma(D)^\perp$, on using Lemma 1 and Equations (10), (11) and (13)–(15), we get

$$\begin{aligned}[U,V] = \quad & \phi(\hat{\nabla}_V P_2U - \hat{\nabla}_U P_2V + T_V\omega U - T_U\omega V + 2Q_VU) \\ & +\omega(\hat{\nabla}_V P_2U - \hat{\nabla}_U P_2V + T_V\omega U - T_U\omega V + 2Q_VU) \\ & +B(T_VP_2U - T_UP_2V + \mathcal{H}\nabla_V\omega U - \mathcal{H}\nabla_U\omega V + 2P_VU) \\ & +C(T_VP_2U - T_UP_2V + \mathcal{H}\nabla_V\omega U - \mathcal{H}\nabla_U\omega V + 2P_VU). \end{aligned} \tag{16}$$

For any $U, V \in \Gamma(D)^\perp \subset \Gamma(kerF_*)$. Since $\Gamma(kerF_*)$ is integrable, therefore $[U,V] \in \Gamma(kerF_*)$. Comparing the vertical part in (16) we get the result. □

On similar lines we prove the following.

Theorem 2. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a proper generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the distribution $\mathcal{D}$ is integrable if and only if*

$$\hat{\nabla}_U P_1V - \hat{\nabla}_V P_1U - 2Q_UV \in \Gamma(\mathcal{D}),$$
$$and \quad T_UP_1V - T_VP_1U - 2P_UV \in \Gamma(\mu),$$

for $U, V \in \Gamma(\mathcal{D})$.

We now study the geometry of the leaves of distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, we have following propositions.

Proposition 1. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the distribution $\mathcal{D}$ defines a totally geodesic foliation in $\mathcal{M}$ if and only if*

(i) $\hat{\nabla}_X P_2 Z + T_X \omega Z - Q_X Z$ *has no component in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D})^{\perp}$.*
(ii) $\hat{\nabla}_X BW + T_X CW - Q_X Z$ *has no component in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma(kerF_*)^{\perp}$.*

Proof. For $X, Y \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D})^{\perp}$, using Equations (1) and (13)–(15) and Lemma 1 we get

$$g_{\mathcal{M}}(\nabla_X Y, Z) = -g_{\mathcal{M}}(\hat{\nabla}_X P_1 Z + T_X \omega Z - Q_X Z, JY). \tag{17}$$

Now, for $X, Y \in \Gamma(\mathcal{D})$ and $W \in \Gamma(kerF_*)^{\perp}$, using again Equations (1) and (13)–(15) and Lemma 1 we get

$$g_{\mathcal{M}}(\nabla_X Y, W) = -g_{\mathcal{M}}(\hat{\nabla}_X BW + T_X CW - Q_X W, JY). \tag{18}$$

From Equations (17) and (18) we get the required result. □

Proposition 2. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliation if and only if*

(i) $\hat{\nabla}_U P_2 V + T_U \omega V - Q_U V = 0,$
(ii) $CT_U P_2 V + C\mathcal{H}\nabla_U \omega V - CP_U V$ *has no components in μ, for $U, V \in \Gamma(\mathcal{D})^{\perp}$.*

From Propositions 1 and 2 we have the following decomposition theorem.

Theorem 3. *LetF : $(\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the fibers are locally product Riemannian manifold of the form $\mathcal{M}_{\mathcal{D}} \times \mathcal{M}_{\mathcal{D}^{\perp}}$ if and only if,*

(i) $\hat{\nabla}_Y P_2 U + T_Y \omega U - Q_Y U = 0$ *for $Y \in \Gamma(kerF_*)$ and $U \in \Gamma(D)^{\perp}$.*
(ii) $CT_U P_2 V + C\mathcal{H}\nabla_U \omega V - CP_U V$ *has no component in μ, $V \in \Gamma(\mathcal{D})^{\perp}$.*
(iii) $\hat{\nabla}_X BW + T_X CW - Q_X Z$ *has no component in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma(kerF_*)^{\perp}$.*
(iv) $\hat{\nabla}_X P_2 Z + T_X \omega Z - Q_X Z$ *has no component in D for $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.*

Proposition 3. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the distribution $(kerF_*)$ defines a totally geodesic foliation in $\mathcal{M}$ if and only if,*

$$T_V \phi W + \mathcal{H}\nabla_V \omega W + P_W V \in \Gamma(\omega \mathcal{D}^{\perp}),$$
$$\text{and} \quad \hat{\nabla}_V \phi W + T_V \omega W + Q_W V \in \Gamma(\mathcal{D}),$$

for any $V, W \in \Gamma(kerF_*)$.

Proposition 4. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map from a naerly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the distribution $(kerF_*)^{\perp}$ defines a totally geodesic foliation in $\mathcal{M}$ if and only if*

$$BA_X BY + B\mathcal{H}\nabla_X CY + BP_Y X = -\phi\hat{\nabla}_X BY - \phi A_X CY - \phi Q_Y X,$$

for $X, Y \in (kerF_*)^{\perp}$.

Proof. Let $X, Y \in (kerF_*)^{\perp}$. Using Equations (11), (12), (14) and (15) and Lemma 1 we get

$$\begin{aligned}\nabla_X Y = \quad & -B(A_X BY + \mathcal{H}\nabla_X CY + P_Y X) \\ & -C(A_X BY + \mathcal{H}\nabla_X CY + P_Y X) \\ & -\phi(\hat{\nabla}_X BY + A_X CY + Q_Y X) \\ & -\omega(\hat{\nabla}_X BY + A_X CY + Q_Y X).\end{aligned} \tag{19}$$

From Eqaution (19) we get the result. □

We recall a Riemannian map with totally umbilical fibers if

$$T_U V = g_{\mathcal{M}}(U, V)H,$$

for all $U, V \in \Gamma(kerF_*)$, where H is the mean curvature vector of the fibers.

We have

Theorem 4. *Let $F : (\mathcal{M}, g_{\mathcal{M}}, J) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ be a generic Riemannian map with totally umbilical fibers from a nearly Kaehler manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then $H \in \Gamma(\omega\mathcal{D})^{\perp}$.*

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