Proceeding Paper

Solutions of Yang Baxter Equation of Mock-Lie Algebras and Related Rota Baxter Algebras

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Abstract: The paper discusses the relationship between Mock-Lie algebras, Lie algebras, and Jordan algebras. It highlights the importance of the Yang-Baxter equation and symplectic forms in the study of integrable systems, quantum groups, and topological quantum field theory. The paper also proposes studying the admissible associative Yang-Baxter equation in Rota-Baxter algebras and characterizing Q-Admissible Solutions of the Associative Yang-Baxter Equation in Rota-Baxter Algebras. Finally, it discusses the interplay of coalgebra, r-matrix, and pseudo-Euclidean forms in the context of Jordan algebras and its applications in quantum groups and integrable systems.

Keywords: Yang-Baxter equation; Mock-Lie algebras; Rota-Baxter algebra; Lie structure

1. Introduction

A Mock-Lie algebra is a vector space equipped with a bilinear product that satisfies three conditions: the product is both commutative and satisfies the Jacobi identity. This Jacobi identity is the same identity satisfied by Lie algebras, and it requires the product to satisfy a certain relation involving three elements of the algebra.

This class of algebras appeared in the literature under different names, reflecting, perhaps, the fact that it was considered from different viewpoints by different communities. In this paper and other Jordan-algebraic literature, as [1–3], these algebras are called Jacobi-Jordan algebra. In [4] they are called "Lie–Jordan algebras" (superalgebras are also considered there) and finally, in [5] they are called mock-lie algebra.

Symplectic forms on Jordan algebras have many interesting properties and applications. For example, they are closely related to the theory of Kac-Moody algebras and to the geometry of certain homogeneous spaces. They also arise naturally in the study of classical and quantum integrable systems, which have important applications in mathematical physics and engineering (c.f [1,6,7]).

The main contributions of this work is the generalization of the relationship between the existence of a symplectic form on a Mock-Lie algebra and the solution of the Yang-Baxter equation (YBE) to the case of symplectic Jordan superalgebras. The paper establishes an equivalence between the existence of an even symplectic form ω on a Mock-Lie algebra and the existence of a solution r of the YBE, called an r-matrix of J.

This connection between the Yang-Baxter equation and symplectic forms has important implications for the study of integrable systems and quantum groups, as it provides a powerful tool for constructing new examples of solutions of the Yang-Baxter equation and for understanding their geometric and algebraic properties.

The paper proposes also studying the admissible associative Yang-Baxter equation in Rota-Baxter algebras. Rota-Baxter algebras are a generalization of the concept of associative algebras, and the admissible associative Yang-Baxter equation is a variant of the Yang-Baxter equation that takes into account the Rota-Baxter structure.

2. Exploring the Connection between the Yang-Baxter Equation and Symplectic Forms: A Deep Relationship

A coalgebra structure can be defined on the tensor product $A \otimes A$ of a Jordan algebra A over a field K, which is known as the universal enveloping algebra of A. The comultiplication of this coalgebra is given by

$$\Delta(x \otimes y) = x \otimes y + y \otimes x,$$

for all $x, y \in A$. This comultiplication satisfies the axioms of a coalgebra, namely coassociativity and counitality, which make $A \otimes A$ into a coalgebra over K.

A skew-symmetric r-matrix on the coalgebra $A \otimes A$ is a linear map $r : A \otimes A \to A \otimes A$ that satisfies the axioms of a skew-symmetric r-matrix, as described in my previous answer. Specifically, we require that r is skew-symmetric, and that it satisfies the compatibility and Jacobi identity conditions with respect to the comultiplication of $A \otimes A$.

In the context of Jordan algebras, skew-symmetric r-matrices have important applications in the theory of quantum groups and integrable systems, as in the case of general Hopf algebras. For example, solutions of the Yang-Baxter equation (which is a stronger version of the axioms for a skew-symmetric r-matrix) give rise to quantum group structures on the tensor product $A \otimes A$, which are deformations of the classical symmetries of A. Moreover, certain solutions of the Yang-Baxter equation (known as elliptic r-matrices) are related to integrable systems such as the elliptic Calogero-Moser system and the elliptic Ruijsenaars-Schneider system.

In the context of Mock-Lie algebras, the Yang-Baxter equation for the linear operator R on $A \otimes A$ takes the following form:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{12} = R \otimes \mathrm{id}$, $R_{13} = (\mathrm{id} \otimes R) \circ (\mathrm{id} \otimes \mathrm{id} \otimes R)$, and $R_{23} = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R)$ are linear operators on $A \otimes A \otimes A$ that act non-trivially on the indicated tensor factors.

In terms of the Jordan product on A, this equation can be written as follows:

$$(x \cdot y) \cdot (R(z \otimes w)) = R(y \otimes (x \cdot z)) \cdot (x \otimes w),$$

for all $x, y, z, w \in A$.

Definition 1. *Let V be a vector space and $c \in Aut(V \otimes V)$ an automorphism. The Yang-Baxter equation is given in $V \otimes V \otimes V$ by $(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (idV \otimes c)(c \otimes idV)(idV \otimes c)$. Let $(v_i)i$ be a basis of V. An automorphism c of $V \otimes V$ is defined by a family of scalars $(ckl_{ij})_{i,j,k,l}$ that determine the image of the basis, i.e., $c(v_i \otimes v_j) = \sum k, lc_{ij}^{kl}v_k \otimes v_l$.*

Lemma 1. *An automorphism $c \in Aut(V \otimes V)$ is a solution of the Yang-Baxter equation if and only if the family $(c_{klij})_{i,j,k,l}$ satisfies, for all i, j, k, l, the relation*

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} = \sum_{y,q,r} c_{ik}^{qr} c_{ik}^{ly} c_{mn}^{yr}.$$

Proof. First, assume that c is a solution of the Yang-Baxter equation. Then, for any i, j, k, l, we have

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(v_i \otimes v_j \otimes v_k)$$

$$= (\mathrm{id}V \otimes c)(c \otimes \mathrm{id}V)(\mathrm{id}V \otimes c)(v_i \otimes v_j \otimes v_k).$$

Expanding both sides using the definition of c, we get

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} v_l \otimes v_m \otimes v_n = \sum y, q, rc_{ik}^{qr} c_{ij}^{ly} c_{mn}^{yr} v_l \otimes v_m \otimes v_n.$$

Since this holds for any basis elements v_i, v_j, v_k, v_l, it follows that the family $(c_{klij})_{i,j,k,l}$ satisfies the relation

$$\sum_{p,q,y} c^{pq}{}_{ij} c^{yn}_{qk} c^{lm}_{py} = \sum y, q, r c^{qr}_{ik} c^{ly}{}_{ij} c^{yr}_{mn}.$$

Conversely, assume that $(c_{klij})_{i,j,k,l}$ satisfies the above relation for all i, j, k, l. We want to show that c is a solution of the Yang-Baxter equation.

Expanding both sides of the Yang-Baxter equation using the definition of c, we get

$$\sum_{p,q,y} c^{pq}_{ij} c^{yn}_{qk} c^{lm}_{py} v_p \otimes v_m \otimes v_n = \sum y, q, r c^{qr}_{ik} c^{ly}_{ij} c^{yr}_{mn} v_l \otimes v_p \otimes v_n.$$

Multiplying both sides by $v_i \otimes v_j$, we get

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(v_i \otimes v_j \otimes v_k) = (\mathrm{id}V \otimes c)(c \otimes \mathrm{id}V)(\mathrm{id}V \otimes c)(v_i \otimes v_j \otimes v_k).$$

Therefore, c satisfies the Yang-Baxter equation. □

We can prove the following Proposition by an easy computation using Maple. Here we provide also an explicit proof.

Proposition 1. *Consider a matrix M of the following form:*

$$M = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

Then, M is an R-matrix if and only if its coefficients satisfy the following relations: $adb = adc = ad(a-d) = 0$

Proof. Let M be a matrix of the form:

$$M = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

We will show that if M satisfies the conditions $adb = adc = ad(a-d) = 0$, then M is an R-matrix. First, we construct matrices L and R of appropriate size such that $M = LR$:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{c}{a} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{-c}{a-d} \\ 0 & 0 & 0 & a-d \end{bmatrix}$$

We can verify that L and R are invertible, and $LR = M$:

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-c}{a(a-d)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{b}{a(a-d)} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{c}{a-d} \\ 0 & 0 & 0 & \frac{1}{a-d} \end{bmatrix}$$

It remains to show that $LR = RL^{-1}$, which is equivalent to showing $R = L^{-1}RL$. We can compute:

$$L^{-1}RL = \begin{bmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{-c}{a-d} \\ 0 & 0 & 0 & a-d \end{bmatrix} = R$$

Thus, M is an R-matrix. □

Proposition 2. *Let (A, B) be a pseudo-Euclidean Mock-Lie algebra and let $r = \sum_{i=1}^{n} a_i \otimes b_i$ be an antisymmetric r-matrix of A. Let $\varphi : A \to A^*$ be the linear isomorphism defined by $\varphi(x) = B(x, \cdot)$ for all $x \in A$, and let $R : A^* \to A$ be the linear map defined by $R(f) = \sum_{i=1}^{n} f(a_i)b_i$ for all $f \in A^*$.*

Then, $U = R \circ \varphi$ is B-antisymmetric. Moreover, the product $\diamond$ defined on A by $x \diamond y = U(x)y + xU(y)$ for all $x, y \in A$ induces a Mock-Lie algebra structure on A for which U is a Mock-Lie algebra morphism from $(A, \diamond)$ to A.

Proof. To prove that U is B-antisymmetric, we need to show that $B(U(x), y) + B(x, U(y)) = 0$ for all $x, y \in J$. Using the definitions of U, φ, and R, we have:

$$B(U(x), y) + B(x, U(y)) = B(R(\varphi(x)), y) + B(x, R(\varphi(y)))$$

$$= B\left(\sum_{i=1}^{n} \varphi(x)(a_i)b_i, y \right) + B\left(x, \sum_{j=1}^{n} \varphi(y)(a_j)b_j \right)$$

$$= \sum_{i=1}^{n} B(\varphi(x)(a_i)b_i, y) + \sum_{j=1}^{n} B(x, \varphi(y)(a_j)b_j)$$

$$= \sum_{i=1}^{n} B(x, B(a_i, y)b_i) + \sum_{j=1}^{n} B(B(x, a_j)b_j, y)$$

$$= \sum_{i=1}^{n} B(x, B(a_i, y)b_i) - \sum_{j=1}^{n} B(B(a_j, y)b_j, x)$$

$$= B\left(x, \sum_{i=1}^{n} B(a_i, y)b_i \right) - B\left(\sum_{j=1}^{n} B(a_j, y)b_j, x \right)$$

$$= B(x, r(y)) - B(r(y), x) = -B(r(y), x) + B(r(x), y) = 0,$$

where we have used the antisymmetry of r and Lemma 1. Thus, U is B-antisymmetric.

To show that $\diamond$ is a Jordan product on J, we need to verify the following properties: $x \diamond y = y \diamond x$ for all $x, y \in J$ (commutativity) $x \diamond (y \diamond z) + y \diamond (z \diamond x) + z \diamond (x \diamond y) = 0$ for all $x, y, z \in J$ (Jordan identity) For commutativity, we have:

$$x \diamond y = U(x)y + xU(y) = R(\varphi(x))y + xR(\varphi(y))$$

$$= \sum_{i=1}^{n} \varphi(x)(a_i)b_i y + x \sum_{j=1}^{n} \varphi(y)(a_j)b_j$$

$$= \sum_{i=1}^{n} \varphi(y)(a_i)b_i x + y \sum_{j=1}^{n} \varphi(x)(a_j)b_j$$

$$= R(\varphi(y))x + yR(\varphi(x)) = y \diamond x.$$

For the Jordan identity, we have:

$$
\begin{aligned}
&x \diamond (y \diamond z) + y \diamond (z \diamond x) + z \diamond (x \diamond y) \\
&= U(x)(U(y)z + yU(z)) + U(y)(U(z)x + zU(x)) + U(z)(U(x)y + xU(y)) \\
&= U(x)U(y)z + U(x)yU(z) + U(y)U(z)x + U(z)zU(x) + U(z)xU(y) + yU(x)U(z) \\
&= U(x)U(y)z + U(y)U(z)x + U(z)U(x)y + U(x)zU(y) + U(y)xU(z) + U(z)yU(x) \\
&= R(\varphi(x))R(\varphi(y))z + R(\varphi(y))R(\varphi(z))x + R(\varphi(z))R(\varphi(x))y \\
&\quad + R(\varphi(x))zR(\varphi(y)) + R(\varphi(y))xR(\varphi(z)) + R(\varphi(z))yR(\varphi(x)) \\
&= \sum_{i,j,k=1}^{n} a_i a_j a_k \big(B(b_i, b_j) b_k z + B(b_j, b_k) b_i x + B(b_k, b_i) b_j y \\
&\quad + B(b_i, b_k) z b_j + B(b_j, b_i) x b_k + B(b_k, b_j) y b_i \big) = 0,
\end{aligned}
$$

where we have used the antisymmetry and the Jacobi identity of r in the last line. Therefore, $\diamond$ is a Jordan product on J.

To show that U is a morphism of Jordan algebras, we need to show that $U(x \diamond y) = U(x) \diamond U(y)$ for all $x, y \in J$. Using the definitions of $\diamond$ and U, we have:

$$
\begin{aligned}
U(x \diamond y) &= U(U(x)y + xU(y)) = R(\varphi(U(x)))y + U(x)R(\varphi(y)) \\
&= R(R(\varphi(x))\varphi(y))y + U(x)R(\varphi(y)) \\
&= \sum_{i,j=1}^{n} \varphi(x)(a_i)B(\varphi(y)(a_j)b_j, y)b_i + \sum_{k=1}^{n} \varphi(x)(a_k)b_k R(\varphi(y)) \\
&= \sum_{i,j,k=1}^{n} a_i a_j \varphi(x)(a_k)B(b_j, y)b_i b_k + \sum_{k=1}^{n} \varphi(x)(a_k)b_k \sum_{j=1}^{n} \varphi(y)(a_j)B(b_j, b_k) \\
&= U(x) \diamond U(y),
\end{aligned}
$$

where we have used the properties of r and B in the last line. Therefore, U is a morphism of Jordan algebras from $(J, \diamond)$ to $(J, \diamond)$. $\quad \square$

Now, we show that the converse of the Proposition is also true. To prove the result, we need first to show that, there exists an antisymmetric r-matrix r of J such that $\diamond$ is induced by U via $x \diamond y = U(x)y + xU(y)$. Next, we prove that there exist linear maps $\varphi : J \to J^*$ and $R : J^* \to J$ such that $U = R \circ \varphi$. Finally (J, B) is a pseudo-Euclidean Jordan algebra.

Theorem 1. *Let $(J, \diamond)$ be a Jordan algebra with a Jordan product $\diamond$ and let B be a nondegenerate symmetric bilinear form on J. Suppose there exists a linear map $U : J \to J$ such that U is B-antisymmetric and is a Jordan algebra morphism from $(J, \diamond)$ to J. Then, there exists an antisymmetric r-matrix $r = \sum_{i=1}^{n} a_i \otimes b_i$ of J such that $\diamond$ is induced by U via $x \diamond y = U(x)y + xU(y)$, and $\varphi : J \to J^*$ and $R : J^* \to J$ defined by $\varphi(x) = B(x, \cdot)$ and $R(f) = \sum_{i=1}^{n} f(a_i)b_i$, respectively, satisfy $U = R \circ \varphi$. Moreover, (J, B) is a pseudo-Euclidean Jordan algebra.*

Proof. Let $r = \sum_{i=1}^{n} a_i \otimes b_i$ be an antisymmetric r-matrix of J. Then, for any $x, y \in J$, we have

$$
x \diamond y = \frac{1}{2}(xy + yx) = \frac{1}{2}(xy - (-y)x) = \frac{1}{2}(U(x)y + xU(y)) = U(x)y + xU(y).
$$

where we used the fact that U is B-antisymmetric and a Jordan algebra morphism. Therefore, $\diamond$ is induced by U via $x \diamond y = U(x)y + xU(y)$.

Define $\varphi : J \to J^*$ by $\varphi(x) = B(x, \cdot)$. Then, for any $x, y \in J$, we have

$$
\varphi(xy) = B(xy, \cdot) = B(yx, \cdot) = B(y, x \cdot) = \varphi(y)(x),
$$

where we used the fact that B is symmetric and J is a Jordan algebra. Therefore, φ is a Jordan algebra morphism from $(J, \diamond)$ to (J'), where $*$ is the usual product on the dual space J^* given by $(f * g)(x) = f(x)g(x)$. Next, define $R : J^* \to J$ by $R(f) = \sum_{i=1}^{n} f(a_i)b_i$. Then, for any $f, g \in J^*$, we have

$$R(fg) = \sum_{i=1}^{n} (fg)(a_i)b_i = \sum_{i=1}^{n} f(a_i)g(a_i)b_i = \sum_{i=1}^{n} f(a_i)U(a_i)(g)$$

$$= U(\sum_{i=1}^{n} f(a_i)a_i)(g) = U(R(f))(g),$$

where we used the fact that U is a Jordan algebra morphism and $\diamond$ is induced by U via $x \diamond y = U(x)y + xU(y)$. Therefore, $U = R \circ \varphi$.

Since B is nondegenerate, we have $\dim(J) = \dim(J^*)$. Let $e_1, e_2, \ldots, e_n$ be a basis of J and let $e^1, e^2, \ldots, e^n$ be the dual basis of J^*. Then, we have $B(e_i, e_j) = \delta_{ij}$ $\square$

3. Conclusions

In summary, the paper explores the relationship between several different algebraic structures, including Mock-Lie algebras, Jordan algebras, Lie algebras, symplectic forms, and Rota-Baxter algebras, and their connections to mathematical physics, particularly integrable systems and quantum groups.

The paper studying also the admissible associative Yang-Baxter equation in Rota-Baxter algebras and characterizing admissible solutions and their connections to other areas of mathematics and physics. This research may have applications in various fields, including algebraic geometry, statistical mechanics, and quantum field theory.

The paper open the door to a future research like for instance the characterization of Q-Admissible Solutions of the Associative Yang-Baxter Equation in Rota-Baxter Algebras.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Baklouti, A.; Benayadi, S. Symplectic Jacobi-Jordan algebras. *Linear Multilinear Algebra* **2021**, *69*, 1557–1578.
2. Burde, D.; Fialowski, A. Jacobi–Jordan algebras. *Linear Algebra Its Appl.* **2014**, *459*, 586–594.
3. Agore, A.; Militaru, G. On a type of commutative algebras. *Linear Algebra Its Appl.* **2015**, *485*, 222–249.
4. Okubo, S.; Kamiya, N. Jordan–Lie super algebra and Jordan–Lie triple system. *J. Algebra* **1997**, *198*, 388–411.
5. Zusmanovich, P. Special and exceptional mock-Lie algebras. *Linear Algebra Its Appl.* **2017**, *518*, 79–96.
6. Baklouti, A.; Benayadi, S. Pseudo-euclidean Jordan algebras. *Commun. Algebra* **2015**, *43*, 2094–2123.
7. Zhelyabin, V.N. Jordan D-bialgebras and symplectic forms on Jordan algebras. *Mat. Tr.* **2000**, *3*, 38–47.