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Proceeding Paper

Some Basic Inequalities on Riemannian Manifolds Equipped with Metallic Structure †

Majid Ali Choudhary

Department of Mathematics, School of Sciences, Maulana Azad National Urdu University, Hyderabad, India; majid_alichoudhary@yahoo.co.in

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Abstract: We construct sharp inequality for submanifolds of metallic Riemannian space forms using generalised normalised δ-Casorati curvatures. In the same ambient space, we also derive the generalised Wintgen inequality for some submanifolds. The equality cases are also covered.

Keywords: slant submanifolds; metallic structure; riemannian manifolds; δ-casorati curvature; wintgen inequality; optimal inequality

MSC: 53B05; 53B20; 53C25; 53C40

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1. Introduction

There has been significant work in the field of golden differential geometry since M. Crasmareanu and C. E. Hretcanu established the golden structure on a Riemannian manifold in 2008 [1]. They also introduced the concept of metallic structure in 2013 as a generalisation of golden structure defined on Riemannian manifolds [2]. Several curvature-related properties of metallic Riemannian manifolds have recently been explored in [3,4].

By generating an obvious inequality, B. Y. Chen developed Chen's invariants (otherwise known as δ-invariants) as a tool to examine the link between intrinsic and extrinsic invariants. The research of Chen invariants and Chen-type inequalities in various submanifolds for numerous ambient spaces ([5–7], etc.) was made possible by this advancement in the field of differential geometry.

In a significant movement, F. Casorati [8] substituted the conventional Gauss curvature with the Casorati curvature, which set the groundwork for the definition of optimal inequalities for submanifolds in various ambient spaces using Casorati curvatures. Many researchers have used Casorati curvatures extensively to find the best inequalities for submanifolds in various ambient spaces [9–11].

Now take a look at Wintgen inequality, which is a sharp geometric inequality involving intrinsic and extrinsic invariants for surface $\mathcal{M}^2$ in E^4. P. Wintgen [12] is credited with finding it using the following equation

$$||\mathcal{H}||^2 \geq \mathcal{K} + |\mathcal{K}^{\perp}|.$$

$\mathcal{K}$ and $\mathcal{K}^{\perp}$ are used to represent the Gauss curvature and normal curvature of $\mathcal{M}^2$, respectively, whereas $\mathcal{H}$ is employed for the mean curvature. Also, when the ellipse of the surface's curvature in Euclidean space looks to be a circle, the equality in the connection mentioned above holds.

The aforementioned inequality was independently researched and improved in the years that followed for surfaces with arbitrary codimension in a real space with constant sectional curvature as

$$||\mathcal{H}||^2 + c \geq \mathcal{K} + |\mathcal{K}^{\perp}|.$$

P. J. De Smet, F. Dillen, L. Verstraelen, and L. Vrancken [13] proposed the generalised Wintgen inequality as a natural generalisation of the aforementioned inequality, and they demonstrated that the following inequality is satisfied at every point of submanifold $\mathcal{M}^n$ of real space form $\overline{\mathcal{M}}^{n+m}(c)$ of constant sectional curvature c

$$\rho \leq ||\mathcal{H}||^2 - \rho^{\perp} + c,$$

where, correspondingly, ρ stands for the normalised scalar curvature and $\rho^{\perp}$ for the normalised normal scalar curvature of $\mathcal{M}$. The DDVV conjecture and the normal scalar curvature conjecture are other names for this hypothesis. It is important to keep in mind that the submanifolds for which the equality in the previous relation holds are referred to as Wintgen ideal submanifolds (see [14]). Although J. Q. Ge, Z. Z. Tang [15], and Z. Lu [16] demonstrated the general case of the DDVV conjecture, respectively. For many submanifolds in various ambient spaces, there has been extensive research and extension of the generalised Wintgen inequality [17–20].

We concentrate this note to proving sharp inequalities involving generalised normalised δ-Casorati curvatures for submanifolds of metallic Riemannian space forms, drawing inspiration from the aforementioned findings. Moreover, generalised Wintgen inequalities for submanifolds form in the same ambient space, and the equality instances are also discussed.

2. Main Results

Theorem 1. *Let $\mathcal{M}$ be an n-dimensional slant submanifold of an m-dimensional locally metallic space form $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \varphi)$. Then*

(i) the generalized normalized δ-Casorati curvature $\delta_c(r; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(r; n-1)}{n(n-1)} + \frac{1}{2}A_1 + \frac{1}{2}A_2 \tag{1}$$

for any real number r, $0 < r < n(n-1)$;

(ii) the generalized normalized δ-Casorati curvature $\widehat{\delta}_c(r; n-1)$ satisfies

$$\rho \quad \leq \quad \frac{\widehat{\delta}_c(r; n-1)}{n(n-1)} + \frac{1}{2}A_1 + \frac{1}{2}A_2 \tag{2}$$

$r > n(n-1)$;

where
$A_1 = \frac{1}{(p^2+4q)}(c_1 + c_2)\left\{p^2 + 2q + \frac{2}{n(n-1)}[tr^2\varphi - (p \cdot trT + nq)\cos^2\theta] - \frac{2p}{n}tr\varphi\right\}$,
$A_2 = \frac{1}{\sqrt{p^2+4q}}(c_1 - c_2)\left(\frac{2}{n}tr\varphi - p\right)$.

In addition, the Equations (1) and (2) also hold for equality if and only if for some orthonormal frame $\{E_1, \ldots, E_n, E_{n+1}, \ldots, E_m\}$, the shape operators A_r, $r \in \{n+1, \ldots, m\}$ take the following forms:

$$S_{n+1} = \begin{pmatrix} f & 0 & 0 & \ldots & 0 & 0 \\ 0 & f & 0 & \ldots & 0 & 0 \\ 0 & 0 & f & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r}f \end{pmatrix}, \qquad S_{n+2} = \cdots = S_m = 0. \tag{3}$$

Theorem 2. *Let $\mathcal{M}$ represent a θ-slant submanifold of dimension n in a locally metallic space form $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \varphi)$. Then, we have*

$$\rho_N \leq ||\mathcal{H}||^2 - 2\rho + A_1 + A_2. \tag{4}$$

Additionally, the Equation (4) holds for equality if and only if for some orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, the shape operator S take the following form:

$$S_{n+1} = \begin{pmatrix} a & d & 0 & \ldots & 0 & 0 \\ d & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, \tag{5}$$

$$S_{n+2} = \begin{pmatrix} b+d & 0 & 0 & \ldots & 0 & 0 \\ 0 & b-d & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, \tag{6}$$

$$S_{n+3} = \begin{pmatrix} c & 0 & 0 & \ldots & 0 & 0 \\ 0 & c & 0 & \ldots & 0 & 0 \\ 0 & 0 & c & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & c & 0 \\ 0 & 0 & 0 & \ldots & 0 & c \end{pmatrix}, \qquad S_{n+4} = \cdots = S_m = 0, \tag{7}$$

where a, b, c and d are real functions on $\mathcal{M}$.

Remark 1. *We can also derive similar inequalities for Riemannian manifolds equipped with the golden structure, the silver structure, the bronze structure, the subtle structure, the copper structure, the nickel structure etc.*

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