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Valuations on Structures More General Than Fields †

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Abstract: Valuation theory is an important area of investigation in algebra with applications in algebraic geometry and number theory. In 1957 M. Krasner introduced hyperfields, which are field-like objects with a multivalued addition, to describe some structures arising naturally from valued fields. We would like to discuss the possibility of generalising the notion of valuation to the multivalued setting and the potential that this higher point of view has in the understanding of classical valuation theory. We will see that a valuation on a field K is nothing but a homomorphism of hyperfields from K onto a special type of hyperfield, which we call (generalised) tropical hyperfield.

Keywords: hyperfield; valuation; valued field; tropical hyperfield

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1. Introduction

Valuation theory has its origins in the study of p-adic numbers' structure arising as the completion of the field of rational numbers with respect to the metrics naturally induced by prime numbers p (see [1]). The explicit arithmetic criteria, available on complete fields, for deciding wether a rational number is representable by a quadratic form, constitute the main reason for the success of Hasse's Local-Global principle [2]. This principle represents one of the first successful applications of valuation theory in the realm of algebraic number theory after the initial efforts of Kürschák [3] and Ostrowski [4–6] in establishing its foundations.

Further work by Ostrowski [7] set the ground for a great development of valuation theory by relating it to Galois theory. However, it is thanks to the ideas illustrated by Krull in [8] that the techniques of valuation theory were able to show their effectiveness in more and more areas of mathematics, other than number theory.

The general definition of valuation on a field formulated by Krull (which is the same that we will adopt below) is fundamentally related to the concept of *place*. The latter is essential for the understanding of many problems in algebraic geometry as resolution of singularities and local uniformization (for more details see e.g., [9] and references therein). Moreover, the breakthrough theorem of Ax–Kochen [10–12] and Ershov [13] have inspired many mathematicians to investigate the model theory of valued fields, which generated a lot of interesting and deep mathematical ideas and constructions.

Even though valuation theory records a lot of success in its history, as often happens in mathematics, many relevant problems remain open (e.g., local uniformization in positive characteristic or decidability of the elementary theory of the Laurent series field over a field of prime cardinality). The challenges that these and other problems offer push us to the limits of the existing theories, were we imagine new approaches with the guidance of the patterns that we are able to recognise and explain.

Let K be a field and Γ a linearly ordered abelian group (always written additively). A surjective map

$$v : K \to \Gamma \cup \{\infty\}$$

is called a *(Krull) valuation on* K if it satisfies the following properties:

V1. $v(x) = \infty$ if and only if $x = 0$,

V2. $v(xy) = v(x) + v(y)$,
V3. $v(x+y) \geq \min\{v(x), v(y)\}$,

for all $x, y \in K$. Here, ∞ is a symbol such that $\gamma + \infty = \infty + \gamma = \infty > \gamma$ for all $\gamma \in \Gamma$. If a valuation v on a field K is given, then (K, v) is usually called a *valued field*. We shall denote Γ by vK and call it the *value group of* (K, v). The *value* $v(x)$ of $x \in K$ will often be written as vx, if there is no risk of confusion. For simplicity, in the sequel all valued fields are assumed to be *non-trivially valued*, i.e., satisfying $vK \neq \{0\}$. Under this assumption, vK, and thus K, are infinite.

If (K, v) is a valued field and $\gamma \geq 0$ is a nonnegative element of the value group vK, then it is well-known that

$$U_v^\gamma := \{x \in K \mid v(x-1) > \gamma\}$$

is a subgroup of $K^\times := K \setminus \{0\}$ with respect to multiplication. It is called the group of *principal units of level* γ in (K, v). Consider the corresponding quotient group:

$$K_\gamma^\times := K^\times / U_v^\gamma = \{xU_v^\gamma \mid x \in K^\times\}$$

and introduce the following notation:

$$xU_v^\gamma \boxplus yU_v^\gamma := \left\{zU_v^\gamma \mid z = x' + y' \text{ for some } x' \in xU_v^\gamma \text{ and } y' \in yU_v^\gamma\right\}.$$

Consider also $0U_v^\gamma = \{0\}$ and set

$$K_\gamma := K_\gamma^\times \cup \{0U_v^\gamma\}.$$

We will sometimes abuse notation and denote the element $0U_v^\gamma$ of K_γ also by 0. Similarly, we will sometimes denote by 1 the element $1U_v^\gamma$ of K_γ. Motivations for these choices will be clarified later. It is natural to extend $\boxplus$ to a function

$$\boxplus : K_\gamma \times K_\gamma \to \mathcal{P}(K_\gamma),$$

where $\mathcal{P}(K_\gamma)$ denotes the power set of K_γ. Such a function is sometimes called a *(binary) multivalued operation* on the set K_γ. Let us extend it further to a binary operation on $\mathcal{P}(K_\gamma)$ as follows: for $A, B \subseteq K_\gamma$ set

$$A \boxplus B := \bigcup_{(a,b) \in A \times B} a \boxplus b,$$

with the understanding that an empty union is equal to the empty set. By direct inspection, it is not difficult to verify that the following properties hold:

1. $a \boxplus b \neq \emptyset$, for all $a, b \in K_\gamma$ (i.e., $\boxplus$ is a *hyperoperation*).
2. $(a \boxplus b) \boxplus c = a \boxplus (b \boxplus c)$ as sets, for all $a, b, c \in K_\gamma$.
3. $a \boxplus b = b \boxplus a$ as sets, for all $a, b \in K_\gamma$.
4. For all $a \in K_\gamma$ there exists a unique $a^- \in K_\gamma$ such that $0 \in a \boxplus a^-$.
5. If $c \in a \boxplus b$, then $a \in c \boxplus b^-$, for all $a, b, c \in K_\gamma$.
6. $K_\gamma \setminus \{0\}$ is an abelian group with respect to some operation $\cdot$, with neutral element 1 and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in K_\gamma$.
7. $c \cdot (a \boxplus b) = c \cdot a \boxplus c \cdot b$ as sets, for all $a, b, c \in K_\gamma$, where $c \cdot (a \boxplus b) = \{c \cdot d \mid d \in a \boxplus b\}$.

Clearly, the operation $\cdot$ in properties 6 and 7 above is the multiplication induced from the multiplicative structure of the field K.

The construction that we have just described was first studied by M. Krasner in the paper [14] (the article is included in Krasner's collected works ([15], pp. 413–490), where, inspired by the above observations, he also gave the following general definition:

Definition 1. *Any structure* $(F, \boxplus, \cdot, 0, 1)$, *where* $\boxplus$ *is a binary multivalued operation on* F *and* $\cdot$ *is a binary operation on* F, *satisfying properties 1–7 above is called a* hyperfield.

The notion of hyperfield thus generalises the notion of field by allowing the additive operation to be multivalued. Any field K is a hyperfield with respect to the hyperoperation $x \boxplus y := \{x + y\}$. Conversely, any hyperfield F whose multivalued operation $\boxplus$ satisfies $|a \boxplus b| = 1$ for all $a, b \in F$ can be naturally regarded as a field. In this short paper, we will discuss the possibility of defining valuations on hyperfields and discuss some new possibilities that the resulting theory offers.

2. Valued Hyperfields

We begin this section with the following result of Krasner, which motivates further the aim of finding a suitable notion of valuation for hyperfields.

Proposition 1 ([14]). *Let* (K, v) *be a valued field and* $\gamma \geq 0$ *be a nonnegative element of* vK. *Then* $vx = 0$ *for all* $x \in U_v^\gamma$. *Moreover, the map*

$$\begin{aligned} v_\gamma : K_\gamma &\to vK \cup \{\infty\} \\ xU_v^\gamma &\mapsto vx \end{aligned}$$

is surjective and satisfies

1. $v_\gamma a = \infty$ *if and only if* $a = 0$, *for all* $a \in K_\gamma$.
2. $v_\gamma(a \cdot b) = v_\gamma a + v_\gamma b$, *for all* $a, b \in K_\gamma$.
3. *If* $c \in a \boxplus b$, *then* $v_\gamma c \geq \min\{v_\gamma a, v_\gamma b\}$, *for all* $a, b, c \in K_\gamma$.
4. *If* $c \in a \boxplus b$, *then* $d \in a \boxplus b$ *if and only if* $v_\gamma e > \gamma + \min\{v_\gamma a, v_\gamma b\}$ *for every* $e \in c \boxplus d^-$, *for all* $a, b \in K_\gamma$.
5. *If* $0 \notin a \boxplus b$, *then* $c, d \in a \boxplus b$ *implies* $v_\gamma c = v_\gamma d$, *for all* $a, b \in K_\gamma$.

Proof. Since in the ordered abelian group vK the operation is compatible with the order and

$$v(1) = v(1 \cdot 1) = v(1) + v(1),$$

$v(1) = 0$ must hold in any valued field (K, v); we deduce that $v(x^{-1}) = -vx$ also holds for all $x \in K^\times$. A similar reasoning yields $v(-1) = 0$ and $v(-x) = vx$ for all $x \in K$ also follows. Moreover, if $vx < vy$, then $v(x + y) = vx$. Indeed, if the latter would not hold, then we would find

$$vx = v((x + y) - y) \geq \min\{v(x + y), v(-y)\} = \min\{v(x + y), vy\} > vx,$$

a contradiction. Let now $x \in U_v^\gamma$, i.e., $v(x - 1) > \gamma$. If $vx > 0 = v(1)$, then $v(x - 1) = v(1) = 0 \leq \gamma$ and if $vx < 0$, then $v(x - 1) = vx < 0 \leq \gamma$. Therefore, $vx = 0$ is necessary for $v(x - 1) > \gamma$ to hold.

As a consequence the map v_γ is well-defined and properties 1 and 2 can be verified directly from the properties V1 and V2 of v. For property 3, assume that

$$zU_v^\gamma \in xU_v^\gamma \boxplus yU_v^\gamma$$

holds for some $x, y, z \in K$. By definition of $\boxplus$, $z = xs + yt$ for some $s, t \in U_v^\gamma$, hence

$$vz = v(xs + yt) \geq \min\{v(xs), v(yt)\} = \min\{vx + vs, vy + vt\} = \min\{vx, vy\}$$

follows by property V3 of v since $vs = vt = 0$. For property 4, note that for all $s, t \in U_v^\gamma$ we have

$$v(s - t) = v(s - 1 - t + 1) \geq \min\{v(s - 1), v(1 - t)\} > \gamma.$$

Now take $x, y \in K$ and assume without loss of generality that $vx \leq vy$. By definition of $\boxplus$, for $z, z' \in K$ we have that

$$zU_v^\gamma, z'U_v^\gamma \in xU_v^\gamma \boxplus yU_v^\gamma$$

if and only if $z = xs + yt$ and $z' = xs' + yt'$ for some $s, t, s', t' \in U_v^\gamma$. We deduce that

$$v(zs'' - z't'') = v(x(ss'' - s't'') + y(ts'' - t't'')) > \gamma + \min\{vx, vy\}$$

holds for all $s'', t'' \in U_v^\gamma$, since $ss'', s't'', ts'', t't'' \in U_v^\gamma$. Conversely, if $v(z - z') > \gamma + \min\{vx, vy\} = \gamma + vx$ holds for some $z, z' \in K$ and

$$zU_v^\gamma \in xU_v^\gamma \boxplus yU_v^\gamma ,$$

i.e., $z = xs + yt$ for some $s, t \in U_v^\gamma$, then

$$v((z - z')x^{-1}) = v(z - z') - vx > \gamma.$$

Therefore, for $s' := 1 + (z - z')x^{-1} \in U_v^\gamma$ we obtain that

$$z' = (1 + s' - s)x + yt.$$

Since $v(s' - s) > \gamma$ as we have already shown above, we deduce that

$$z'U_v^\gamma \in xU_v^\gamma \boxplus yU_v^\gamma .$$

It remains to show that property 5 holds as well. For take $x, y \in K$ and assume that

$$v(xt + ys) < v(xt' + ys')$$

for some $s, t, s', t' \in U_v^\gamma$. By property 4, we obtain that

$$v(0 - (xs + yt)) = v(xs + yt) = v(xt + ys - (xt' + ys')) > \gamma + \min\{vx, vy\}.$$

By another application of property 4, we obtain that $0 \in xU_v^\gamma \boxplus yU_v^\gamma$. □

The similarity between properties 1, 2 and 3 and the properties V1, V2 and V3 motivate the following definition.

Definition 2. *If F is a hyperfield, Γ an ordered abelian group and $v : F \to \Gamma \cup \{\infty\}$ a surjective map satisfying properties 1, 2 and 3 of Proposition 1 above, then we call v a* valuation *and (F, v) a* valued hyperfield*. We retain our notation and terminology as explained in the introduction for valuations on fields.*

Krasner as well studied valued hyperfields, but his definition was more restrictive than the above one: he postulated that valuations on hyperfields additionally satisfy property 4 (where the free variable γ is bounded existentially and required to be nonnegative in the value group) and property 5. Even though Krasner's definition captures some relatively well-behaved structures (cf. e.g., [16–19]), our choice can be motivated by the observation of many natural examples which do not fit Krasner's definition. Among these examples we present an important one below.

Example 1 (Generalised tropical hyperfield)**.** *Let Γ be an ordered abelian group and ∞ be a symbol such that $\gamma + \infty = \infty + \gamma = \infty > \gamma$ for all $\gamma \in \Gamma$. For $\gamma, \delta \in \Gamma \cup \{\infty\}$ such that $\gamma \leq \delta$, let us denote by $[\gamma, \delta]$ the set consisting of all $\varepsilon \in \Gamma \cup \{\infty\}$ such that $\gamma \leq \varepsilon \leq \delta$. We consider the multivalued operation $\boxplus$ defined on $\mathcal{T}(\Gamma) := \Gamma \cup \{\infty\}$ as follows:*

$$\gamma \boxplus \infty = \infty \boxplus \gamma = \{\gamma\} \qquad (\gamma \in \mathcal{T}(\Gamma))$$

and

$$\gamma \boxplus \delta := \begin{cases} \{\min\{\gamma,\delta\}\} & \text{if } \gamma \neq \delta, \\ [\gamma,\infty] & \text{otherwise.} \end{cases} \qquad (\gamma,\delta \in \Gamma)$$

It is not difficult to check that $\mathcal{T}(\Gamma)$ is a hyperfield, where the multiplication is given by the operation $+$ of Γ. The hyperfield $\mathcal{T}(\mathbb{R},+,0,>)$, where $<$ denotes the standard order of the real numbers, is known as the tropical hyperfield (see e.g., Section 1 of [20]). We call the hyperfields of the form $\mathcal{T}(\Gamma)$ generalised tropical hyperfields*.*

The identity map on $\mathcal{T}(\Gamma)$ is a valuation on $\mathcal{T}(\Gamma)$ as it (almost trivially) satisfies properties 1–3 of Proposition 1. Nevertheless, property 4 does not hold in this case as $0 \in [0,\infty] = 0 \boxplus 0$, but for all $\delta > 0$ we have that $0 \boxplus \delta = \{0\}$ and $0 > \gamma + 0$ does not hold for any $\gamma \geq 0$.

3. Homomorphisms of Hyperfields

Like other algebraic structures, hyperfields can be arranged in a category. The standard choice for arrows in the category of hyperfields are the homomoprhisms defined as follows:

Definition 3. *Let F_1 and F_2 be hyperfields. A map $\sigma : F_1 \to F_2$ is called a* homomorphism *of hyperfields if the following properties hold:*

H1. $\sigma(a) = 0_2$ *if and only if* $a = 0_1$, *for all* $a \in F_1$.
H2. $\sigma : F_1^\times \to F_2^\times$ *is a homomorphism of groups.*
H3. $\sigma(a \boxplus_1 b) \subseteq \sigma(a) \boxplus_2 \sigma(b)$, *for all* $a, b \in F_1$.

The next observation provides an alternative definition of valuation on hyperfields and in particular on fields (cf. Example 1.8(2) of [20]). We leave the straightforward proof to the reader.

Proposition 2. *Let F be a hyperfield and Γ an ordered abelian group. Then (F, v) is a valued hyperfield with $vF = \Gamma$ if and only if the map $v : F \to \mathcal{T}(\Gamma)$ is a surjective homomorphism of hyperfields.*

Thus, if we think of valuations as surjective homomorphisms of hyperfields, then their properties can be reflected directly into the additive structure of the target hyperfield.

In [21] generalised tropical hyperfields are characterised as hyperfields $(T, \boxplus, \cdot, 0, 1)$ satisfying the following properties:

(T1) for all $a, b \in T$ if $0 \notin a \boxplus b$, then $|a \boxplus b| = 1$, i.e., T is *stringent*.
(T2) $0, 1 \in 1 \boxplus 1$.

It is interesting to note that many generalisations of the notion of valuation that appeared in the literature can analogously be understood as homomorphism of hyperfields and, under this interpretation, correspond to less restrictive properties for their target hyperfield than (T1) and (T2). Below, we briefly present an example.

Example 2. *Let L be a lattice which is also a group with respect to a compatible operation $+$ (i.e., L is an ℓ-group). Let ∞ be a new symbol such that $l + \infty = \infty + l = \infty > l$ for all $l \in L$ and define on $L \cup \{\infty\}$ the following multivalued operation:*

$$l \boxplus \infty = \infty \boxplus l = \{l\} \qquad (l \in L \cup \{\infty\})$$

and

$$n \boxplus m := \{l \in L \mid n \wedge l = l \wedge m = n \wedge m\} \quad (n, m \in L).$$

By some results of Nakano (see Theorem 1 of [22]) it follows that the resulting structure $\mathcal{N}(L)$ is a hyperfield (notice that since any ℓ-group is distributive, L is modular). We have that $\mathcal{N}(\Gamma)$ is stringent if and only if the order of L is linear, in which case L is an ordered abelian group and $\mathcal{N}(\Gamma) = \mathcal{T}(\Gamma)$, as the reader can easily check. Now, it is not difficult to see that v is a lattice-valuation of a field K onto L (in the sense of e.g., [23]) if and only if v is a surjective homomorphism *of hyperfields $K \to \mathcal{N}(L)$. Thus, in this case it is property (T1) of the target hyperfield that is relaxed, while (T2) holds in $\mathcal{N}(L)$.*

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