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Dynamics of Holling Type II Eco-Epidemiological Model with Fear Effect, Prey Refuge and Prey Harvesting †

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Abstract: This paper is investigating the prey-predator model, which includes a fear effect on susceptible prey through infected prey. Also, the predator consumes its prey in the form of Holling-type interactions. For the model, first we analyse the existence and local stability of possible non-negative equilibrium points. Further, we examine the Hopf-bifurcation analysis for the corresponding proposed model in the presence of the fear effect. Finally, we demonstrate some numerical simulation results to illustrate our main analytical findings.

Keywords: fear effect; prey refuge; prey harvesting; equilibrium points; stability; Hopf-bifurcation

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1. Introduction

The first predator-prey model was developed and analyzed by Alfred James Lotka and Vito Volterra [1–3]. Anderson and May witnessed prey infection and infection-related instability in their prey-predator model. Numerous investigations of predator-prey behaviors have been conducted in the preceding ten years, compensating for the consequences of various biological variables. In the topic of epidemiological studies, numerous statistical models that contribute for various kinds of prevalence rate and illnesses have been suggested and examined. Anderson and May established the first connection between these two systems [4,5]. The term "eco-epidemiology" was first applied to these models by Chattopadhyay and Arino [5,6].

The prey refuge and harvesting are incorporated into the eco epidemiological model using dynamical behavior without the fear effect has been studied by many authors [1]. To address this lacuna, in this paper, We analyze the Holling type II eco-epidemiology model's behaviour in response to the prey refuge, fear effect, and prey harvesting. In this paper, we examine how the effect of fear is induced by infected prey on susceptible prey.

2. Model Formation

The non-linear differential equation are

$$\left.\begin{aligned} \frac{dS}{dT} &= \frac{r_1 S}{1+LI}\left(1-\frac{S+I}{K}\right) - \eta IS - \frac{\alpha_1 SW}{\beta_1+S} - H_1E_1S, \\ \frac{dI}{dT} &= \eta IS - d_1 I - \frac{\gamma_1(1-\theta)IW}{\beta_1+(1-\theta)I} - H_2E_2I, \\ \frac{dW}{dT} &= -d_2W + \frac{c\gamma_1(1-\theta)IW}{\beta_1+(1-\theta)I} + \frac{c\alpha_1 SW}{\beta_1+S}. \end{aligned}\right\} \quad (1)$$

Table 1. The biological meanings of parameter are in Table.

Parameters	Biological Representation
S	Susceptible Prey
I	Infected Prey
W	Predator
L	Fear effect
r_1	Prey growth rate
k	Environment carrying capacity
β_1	Constant of Half-saturation
α_1	Susceptible Prey rate of Predation
γ_1	Infected Prey Predation rate
c	coefficient of transformation from prey towards predator.
d_1	Death rate of infected prey
d_2	Death rate for predator population
η	The contamination rate for prey
θ	The constant for refuge in prey and predator
H_1	The prey's suspectible coefficient under contagious.
H_2	The prey's infected coefficient under contagious
E	The Harvesting effect of predator

Reduce parameter is as follows: $s = \frac{S}{K}, i = \frac{I}{K}, w = \frac{W}{K}, l = Lk, t = \eta kT$. Now the system becomes,

$$\left.\begin{array}{rcl} \frac{ds}{dt} &=& \frac{rs}{1+li}(1-s-i) - si - \frac{\alpha sw}{\beta+s} - h_1 s, \\ \frac{di}{dt} &=& is - di - \frac{\gamma(1-\theta)iw}{\beta+(1-\theta)i} - h_2 i, \\ \frac{dw}{dt} &=& -\omega w + \frac{c\gamma(1-\theta)iw}{\beta+(1-\theta)i} + \frac{c\alpha sw}{\beta+s}. \end{array}\right\} \tag{2}$$

where, $r = \frac{r_1}{\eta K}, \alpha = \frac{\alpha_1}{\eta K}, h_1 = \frac{H_1 E_1}{\eta K} d = \frac{d_1}{\eta K}, h_2 = \frac{H_2 E_2}{\eta K}, \gamma = \frac{\gamma_1}{\eta K}\beta = \frac{\beta_1}{K}, \omega = \frac{d_2}{\eta K}$ along with the initial conditions $s(0) \geq 0, i(0) \geq 0$, and $w(0) \geq 0$. The above defined functions are in $\mathbb{R}^3_+$.

3. Equilibrium Points

In this section, solutions for the four equilibrium points (2) are discussed.

- The $E_0(0,0,0)$ be the trivial equilibrium point.
- The point of equilibrium between infected prey and predator $E_1(\bar{s},0,0)$, where $\bar{s} = (\frac{r-h_1}{r})$, E_1 always exist provided $h_1 < r$.
- Disease free equilibrium point $E_2(\bar{s},0,\bar{w})$, where $\bar{s} = \frac{\beta\omega}{c\alpha-\omega}$, $\bar{w} = \frac{\beta c((c\alpha-\omega)(r-h_1)-\beta r\omega)}{(c\alpha-\omega)^2}$ E_2 exist for $c\alpha > \omega$ and $h_1 < r(1-\frac{\beta r\omega}{c\alpha<\omega})$.
- Positive equilibrium is $E^*(s^*,i^*,w^*)$, where
  $i^* = \frac{\beta(\beta\omega+(\omega-c\alpha)s^*)}{(1-\theta)(c\alpha s^*+(c\gamma-\omega)(\beta+s^*))}$, $w^* = \frac{\beta c(s^*-d-h_2)(\beta+s^*)}{(1-\theta)(c\alpha s^*+(c\gamma-\omega)(\beta+s^*))}$,
  and the s* is the only positive value that a quadratic solution
  $As^3 - Bs^2 - Cs - D = 0$, where $A = re_4^2$,
  $B = re_4e_5 - re_4e_{10} - \beta\omega e_7 - h_1e_4e_7$,
  $D = r\beta e_6e_{10} - \beta^2 e_8e_9 - h_1\beta e_8e_{10}$,
  $C = r\beta e_4e_6 - re_5e_{10} - \beta e_7e_9 - \beta^2\omega e_8 - h_1\beta e_4e_8 - h_1e_7e_{10}$,
  $e_1 = 1-\theta, e_2 = c\gamma - \omega$,
  $e_3 = \omega - c\alpha, e_4 = e_1(c\alpha + e_2)$,
  $e_5 = e_4 - \beta(e_2+e_3), e_6 = e_1e_2 - \beta\omega$,
  $e_7 = e_4 + l\beta e_3, e_8 = e_1e_2 - l\beta\omega$,
  $e_9 = \omega - c\alpha(d+h_2), e_{10} = \beta e_1e_2$.

4. Local Stability Analysis

It is necessary to calculate the Jacobian matrix, which is provided by, in order to evaluate the stability characteristics of the system (2), $J(E) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$ where

$$u_{11} = \frac{r(1-2s)}{1+li} - i(\frac{r}{1+li}+1) - \frac{\alpha\beta w}{(\beta+s)^2} - h_1, u_{12} = \frac{-r(1+l(1-s))}{(1+li)^2} - s,$$
$$u_{13} = -\frac{\alpha s}{\beta+s}, u_{21} = i, u_{22} = s - d - h_2 - \frac{\beta\gamma(1-\theta)w}{(\beta+(1-\theta)i)^2}, u_{23} = -\frac{\gamma(1-\theta)i}{(\beta+(1-\theta)i)},$$
$$u_{31} = \frac{\beta c\alpha w}{(\beta+s)^2}, u_{32} = \frac{\beta c\gamma(1-\theta)p}{(\beta+i(1-\theta))^2}, u_{33} = -\omega + \frac{c\gamma(1-\theta)i}{\beta+(1-\theta)i} + \frac{\alpha cs}{\beta+s}.$$

Theorem 1. *The point $E_0(0,0,0)$ is stable then it is said to be trivial if $r < h_1$, otherwise unstable.*

Proof. At the point $E_0(0,0,0)$ for the Jacobian matrix, $J(E_0) = \begin{pmatrix} r-h_1 & -r(1+l) & 0 \\ 0 & -d-h_2 & 0 \\ 0 & 0 & -\omega \end{pmatrix}$

Here, the defining equation of the aforementioned Jacobian matrix is, $((r-h_1) - \lambda_{01})((-d-h_2) - \lambda_{02})(-\omega - \lambda_{03}) = 0$, $\lambda_{01} = r - h_1, \lambda_{02} = -d - h_2, \lambda_{03} = -\omega$, here, $\lambda_{02} < 0, \lambda_{03} < 0$ then the equilibrium point E_0 is stable if $r < h_1$, otherwise unstable. □

Theorem 2. *The point of equilibrium between infected prey and predator $E_1(\bar{s},0,0)$, $\bar{s} = (\frac{r-h_1}{r})$ is stable if $\frac{c\alpha}{r\beta+r-h_1}(r-h_1) < \omega$ and $h_1 > r(1-d-h_2)$, otherwise unstable.*

Proof. The Jacobian matrix for E_1 is $J(E_1) = \begin{pmatrix} h_1 - r & -1-r-h_1(l-\frac{1}{r}) & \frac{-\alpha(r-h_1)}{r\beta+r-h_1} \\ 0 & 1-d-h_2-\frac{h_1}{r} & 0 \\ 0 & 0 & \frac{c\alpha(r-h_1)}{r\beta+r-h_1} - \omega \end{pmatrix}$

Here, the characteristic equation of the above Jacobian matrix is, $(h_1 - r - \lambda_{11})(1 - d - h_2 - \frac{h_1}{r} - \lambda_{12})(\frac{c\alpha}{r\beta+r-h_1}(r-h_1) - \omega - \lambda_{13}) = 0$, $\lambda_{11} = h_1 - r, \lambda_{12} = 1 - d - h_2 - \frac{h_1}{r}, \lambda_{13} = \frac{c\alpha}{r\beta+r-h_1}(r-h_1) - \omega$, here, when infection is advanced enough for both predators and prey to be equally successful $E_1(\frac{r-h_1}{r},0,0)$is stable if $\frac{c\alpha}{r\beta+r-h_1}(r-h_1) < \omega$, and $h_1 > r(1-d-h_2)$, otherwise unstable. □

Theorem 3. *The disease free equlibrium point $E_2(\frac{a\omega}{c\alpha-\omega}, 0, \frac{ac((c\alpha-\omega)(r-h_1)-ar\omega)}{(c\alpha-\omega)^2})$ is locally asymptotically stable if $(\frac{a\omega}{c\alpha-\omega} - d) < h_1$ and $r(1-\frac{2a\omega}{c\alpha-\omega}) < h_2$.*

Proof. The Jacobian matrix at E_2 is $J(E_2) = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$ where,

$$v_{11} = r - h_1 - \frac{2\beta r\omega}{c\alpha-\omega} - \frac{(c\alpha-\omega)^2\bar{w}}{\beta\alpha c^2}, v_{12} = \frac{-r(c\alpha-\omega)(1+l)+\beta\omega(rl-1)}{c\alpha-\omega}, v_{13} = -\frac{\delta}{c},$$
$$v_{21} = 0, v_{22} = -d - h_2 + \frac{\beta\omega}{c\alpha-\omega} - \frac{b(1-\theta)\bar{w}}{\beta}, v_{23} = 0, v_{31} = \frac{(c\alpha-\omega)^2\bar{w}}{\beta c\alpha}, v_{32} = \frac{cb(1-\theta)\bar{w}}{\beta},$$
$$v_{33} = 0.$$

Here the characteristic equation of the above Jacobian matrix is $\lambda^3 + P\lambda^2 + Q\lambda + R = 0$, here $P = -v_{11} - v_{22}, Q = -v_{31}v_{13} + v_{22}v_{11}, R = v_{13}v_{22}v_{31}$.

The Routh-Hurwitz conditions state that all of the roots of the aforementioned attribute have negative real portions unless P, R, and $PQ - R$ are positive.

$PQ - R = -v_{11}v_{22}(v_{11} + v_{22}) + v_{11}v_{13}v_{31}$. The sufficient conditions for v_{11} and v_{22} to be negative are $(\frac{a\omega}{c\alpha-\omega} - d) < h_1, r(1 - \frac{2a\omega}{c\alpha-\omega}) < h_2$. The $E_2(\frac{a\omega}{c\alpha-\omega}, 0, \frac{ac((c\alpha-\omega)(r-h_1)-ar\omega)}{(c\alpha-\omega)^2})$ is the region is asymptotically stable on a local level if the theorem's aforementioned condition is satisfied. □

Theorem 4. *The positive equilibrium point E^* is stable locally and exhibits asymptotic stability. If $C > 0, G > 0$ and $CD - G > 0$ where $C = -x_{11} - x_{22}, D = -x_{21}x_{12} + x_{22}x_{11} - x_{13}x_{31} - x_{23}x_{32}, G = x_{13}(x_{22}x_{31} - x_{21}x_{32}) + x_{23}(x_{11}x_{32} - x_{12}x_{31})$.*

Proof. The Jacobian matrix at E^* is $J(E^*) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

where,

$$x_{11} = \frac{r(1-2s^*)}{1+li^*} - i(\frac{r}{1+li^*} + 1) - \frac{\alpha\beta w^*}{(\beta+s^*)^2} - h_1, x_{12} = \frac{-r(1+l(1-s^*))}{(1+li^*)^2} - s^*,$$
$$x_{13} = -\frac{\alpha s^*}{\beta+s^*}, x_{21} = i^*, x_{22} = s^* - d - h_2 - \frac{(s^*-d-h_2)(\beta+s^*)(c\gamma-\omega) + c\alpha s^*}{c\gamma(\beta+s^*)},$$
$$x_{23} = -\frac{\gamma(1-\theta)i^*}{(\beta+(1-\theta)i^*)}x_{31} = \frac{\beta c\alpha p^*}{(\beta+s^*)^2}, x_{32} = \frac{\beta c\gamma(1-\theta)p^*}{(\beta+(1-\theta)i^*)^2}, x_{33} = 0.$$

Here, The Jacobian matrix's distinctive expression is,

$$\lambda^3 + C\lambda^2 + D\lambda + G = 0 \tag{3}$$

here $C = -x_{11} - x_{22}$, $D = -x_{21}x_{12} + x_{22}x_{11} - x_{13}x_{31} - x_{23}x_{32}$ and $G = x_{13}(x_{22}x_{31} - x_{21}x_{32}) + x_{23}(x_{11}x_{32} - x_{12}x_{31})$.

If $C > 0, G > 0, CD - G > 0$. If and solely if C, G and $CD - G$ are positive, all the roots of the aforementioned feature will meet the Routh-Hurwitz requirements and have negative discrete components. Locally, the $E*$is asymptotically stable. □

5. Hopf-Bifurcation Analysis

Throughout this section, we analyse the split of the model in relation to the effect of collecting l. By using the bifurcating constant l and the following theorem, it is demonstrated that there is a Hope-bifurcation [7].

Theorem 5. *The Hopf-bifurcation occurs in the model (2) if the bifurcation parameter l increases above a cutoff value. At $l = l*$, the resulting hope-bifurcation conditions occur.,*
1.$N_1(l^*)N(l^*) - N_3(l^*) = 0$.
2.$\frac{d}{dl}(Re(\lambda(l)))|_{l=l^*} \neq 0$
where, λ is the nought of the characteristic equation related to the internal equilibrium point.

Proof. For $l = l^*$, let the characteristic Equation (3) is in the form

$$(\lambda^2(l^*) + N_2(l^*))(\lambda(l^*) + N_1(l^*)) = 0. \tag{4}$$

Which implies that $\pm i\sqrt{N_2(l^*)}$ and $-N_1(l^*)$ be the roots of the above equation. To attain the Hopf-bifurcation at $l = l^*$, we must complete the following transversality condition.

$$\frac{d}{dl^*}(Re(\lambda(l^*)))| \neq 0.$$

For all l , the general roots of the above Equation (4)

$$\lambda_1 = r(l) + is(l), \lambda_2 = r(l) - is(l), \lambda_3 = -N_1(l).$$

Now, we check the condition $\frac{d}{dl^*}(Re(\lambda(l^*)))| \neq 0$.
Let $\lambda_1 = r(l) + is(l)$ in the (4), we get

$$\mathcal{A}(l) + i\mathcal{B}(l) = 0.$$

where,

$$\mathcal{A}(l) = r^3(l) + r^2(l)N_1(l) - 3r(l)s^2(l) - s^2(l)N_1(l) + N_2(l)r(l) + N_1(l)N_2(l),$$
$$\mathcal{B}(l) = N_2(l)s(l) + 2r(l)s(l)N_1(l) + 3r^2(l)N(l) + s^3(l).$$

In order to fulfill the (4) we must have $\mathcal{A}(l) = 0$ and $\mathcal{B}(l) = 0$, then differentiating $\mathcal{A}$ and $\mathcal{B}$ with respect to l. We have

$$\frac{d\mathcal{A}}{dl} = \chi_1(l)r^{'}(l) - \chi_2(l)s^{'}(l) + \chi_3(l) = 0, \tag{5}$$

$$\frac{d\mathcal{B}}{dl} = \chi_2(l)r^{'}(l) + \chi_1(l)s^{'}(l) + \chi_4(l) = 0. \tag{6}$$

where,

$$\begin{aligned}
\chi_1 &= 3r^2(l) + 2r(l)N_1(l) - 3s^2(l) + N_2(l),\\
\chi_2 &= 6r(l)s(l) + 2s(l)N_1(l),\\
\chi_3 &= r^2(l)N_1^{'}(l) + s^2(l)N_1^{'}(l) + N_2^{'}(l)r(l),\\
\chi_4 &= N_2^{'}(l)s(l) + 2r(l)s(l)N_1^{'}(l).
\end{aligned}$$

On multiplying (5) by $\chi_1(l)$ and (6) by $\chi_2(l)$ respectively

$$r(l)^{'} = -\frac{\chi_1(l)\chi_3(l) + \chi_2(l)\chi_4(l)}{\chi_1{}^2(l) + \chi_2{}^2(l)}. \tag{7}$$

Substituting $r(l) = 0$ and $s(l) = \sqrt{N_2(l)}$ at $l = l^*$ on $\chi_1(l), \chi_2(l), \chi_3(l)$, and $\chi_4(l)$, we obtain

$$\begin{aligned}
\chi_1(l^*) &= -2N_2(h_2^*),\\
\chi_2(l^*) &= 2N_1(l^*)\sqrt{N_2(l^*)}\\
\chi_3(l^*) &= N_3^{'}(l^*) - N_2(l^*)N_1^{'}(l^*),\\
\chi_4(l^*) &= N_2^{'}(l^*)\sqrt{N_2 l^*}.
\end{aligned}$$

The Equation (7), implies

$$r^{'}(l^*) = \frac{N_3^{'}(l^*) - (N_1(l^* N_2(l^*)))}{2(N_2(l^*) + N_1^2(l^*))}, \tag{8}$$

if $N_3^{'}(l^*) - (N_1(l^*)N_2(l^*))^{'} \neq 0$ which implies that $\frac{d}{dl^*}(Re(\lambda(l^*)))| \neq 0$, and $\lambda_3(l^*) = -N_1(l^*) \neq 0$. Therefore the condition $N_3^{'}(l^*) - (N_1(l^*)N_2(l^*))^{'} \neq 0$ is guaranteed that the transversality condition hold, thus the model (2) has enter into the Hopf-bifurcation at $l = l^*$ □

6. Numerical Simulations

Table 2. Numerous numerical models of the system (2) are carried out in this part in order to validate the results of theoretic analyses. The important factors in the current research are the rate of gathering h and the rate of predation, which will be used as control parameters [1,5]. The numerical simulation is carried out using MATLAB software package for the set of parameter values given.

Parameters	Numeric Value
θ	0.2
r	0.5
l	0.5
β	0.2
α	0.32
d	0.1
h_1	0.089
c	0.5
γ	0.13
ω	0.12
h_2	0.15

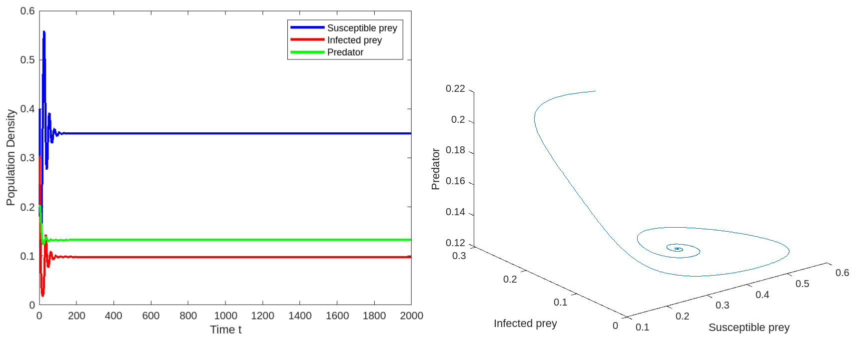

Figure 1.

7. Conclusions

The fear effect, prey refuges, and prey harvesting have all been addressed and examined in this paper's study of a Holling type II eco-epidemiology model. The infected and weakened prey is consumed by the predator. This system has three equilibrium points and one endemic equilibrium point. Local stability and Hopf bifurcation for the suggested models are systematically studied. We have presented numerical simulations to support some key findings. **Author Contributions:**

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