Proceeding Paper

Trigonometrically Fitted Improved Hybrid Method for Oscillatory Problems

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Abstract: Presented in this paper is a trigonometrically fitted scheme based on a class of improved hybrid method for the numerical integration of oscillatory problems. The trigonometric conditions are constructed through which a third algebraic order scheme is derived. Numerical properties of the scheme are analysed. Numerical experiment is conducted to validate the scheme. Results obtained reveal the superiority of the scheme over its equals in the literature

Keywords: oscillatory solution; numerical scheme; trigonometrically fitted, hybrid method; trigonometric conditions; oscillatory problem.

1. Introduction

Our interest in this paper is on the solution of a special class of second order ordinary differential equations (ODEs) whose solution exhibits oscillatory behaviors. In short, the equation together with its boundary conditions (initial value problem (IVP)) takes the following form:

$$y''(x) = f(x, y(x)), y(x_0) = y_0, y'(x_0) = y'_0. \tag{1}$$

It is a special case of second ODEs because the right-hand-side of the main equation is independent of y' component. Over the years, researchers' interest on this particular problem (1) has grown. This is largely due to its applicability in a number of areas in applied sciences including engineering, celestial mechanics, orbital mechanics, chemical kinetics, astrophysics, chemistry, physics and elsewhere [1–12]. Unfortunately, as important as the problem (1), only a few of them could be solved analytically. Hence, the need for numerical schemes.

Traditional numerical schemes like Runge-Kutta methods, Runge-Kutta-Nyström methods, linear multistep method e.t.c for solving second order ODEs could solved (1) only with little accuracy and efficiency due to the behaviours of the solution. Research has shown that an adapted form of the traditional schemes could solve (1) with reduced error and better efficiency [5].

Recently, [11,12] introduced in the literature a new numerical scheme that proved to be more promising in tackling (1). The methods are developed to be implemented in constant coefficients fashion. The method could perform better if adapted to specifically handle (1). This is the main motivation of this paper.

The remaining part of the paper is organized as follows: in Section 2, the proposed scheme is derived; results of numerical experiment are presented in Section 3; discussion of the results is presented in Section 4 and finally, conclusion is given in Section 5.

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2. The Scheme

The general form of improved hybrid method is

$$
\begin{aligned}
y_{n+1} &= \frac{3}{2}y_n - \frac{1}{2}y_{n-2} + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i), \\
Y_i &= \frac{1}{2}(2 + c_i)y_n - \frac{1}{2}c_i y_{n-2} + h^2 \sum_{i=j}^{s} a_{i,j} f(x_n + c_j h, Y_j),
\end{aligned} \tag{2}
$$

where y_{n+1} and y_{n-2} are approximations for $y(x_{n+1})$ and $y(x_{n-2})$, respectively. $a_{i,j}$, b_i and c_i are coefficients of the method and they are real numbers. $i = 1, ..., s$ and $i > j$, because the method is explicit. The coefficients can be summarized as follows:

Table 1. General coefficients of the scheme.

-2	0				
0	0	0			
c_3	a_{31}	a_{32}	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
c_m	a_{m1}	a_{m2}	$\cdots$	a_{mm-1}	0
	b_1	b_2	$\cdots$	b_{m-1}	b_m

2.1. Order condition of the scheme

Algebraic order condition of a method or scheme is a set of equations that causes the successive terms in the Taylor series expansion of local truncation error of the method to vanish. The order conditions of the scheme as derived and presented in [11,12] can be seen in the table below:

Table 2. Order Conditions

t	$\rho(t)$	Order condition
τ	0	-
τ_1	1	-
τ_2	2	$\sum b_i = \frac{3}{2}$
$t_{3,1}$	3	$\sum b_i c_i = -\frac{1}{2}$
$t_{4,1}$	4	$\sum b_i c_i^2 = \frac{3}{4}$
$t_{4,2}$		$\sum b_i a_{i,j} = -\frac{1}{8}$
$t_{5,1}$	5	$\sum b_i c_i^3 = -\frac{3}{4}$
$t_{5,2}$		$\sum b_i c_i a_{i,j} = \frac{3}{8}$
$t_{5,3}$		$\sum b_i a_{i,j} c_j = \frac{5}{24}$
$t_{6,1}$	6	$\sum b_i c_i^4 = \frac{11}{10}$
$t_{6,2}$		$\sum b_i c_i^2 a_{i,j} = \frac{11}{20}$
$t_{6,3}$		$\sum b_i c_i a_{i,j} c_j = \frac{41}{60}$
$t_{6,4}$		$\sum b_i a_{i,j} a_{i,k} = \frac{3}{16}$
$t_{6,5}$		$\sum b_i a_{i,j} c_j^2 = -\frac{87}{360}$
$t_{6,6}$		$\sum b_i a_{i,j} a_{j,k} = \frac{21}{240}$

2.2. Trigonometric Conditions

Suppose we apply the scheme (2) to solve problem (1) whose solution is a linear combination of $\{x^j \exp(\alpha x), x^j \exp(-\alpha x)\}$, exactly, where α is real or complex. But here, we are interested in the complex value. Assuming the solution is exp $(i\alpha x)$, where i is imaginary. Then the trigonometric conditions are obtained as follows:

$$\cos(z) - \frac{3}{2} + \frac{1}{2}\cos(2z) + z^2 \sum_{k=1}^{s} b_k \cos(c_k z) = 0,$$
$$\sin(z) - \frac{1}{2}\sin(2z) + z^2 \sum_{k=1}^{s} b_k \sin(c_k z) = 0,$$
$$\cos(c_i z) - 1 - \frac{1}{2}c_i + \frac{1}{2}c_i\cos(2z) + z^2 \sum_{j=1}^{i-1} a_{ij}\cos(c_j z) = 0,$$
$$\sin(c_i z) - \frac{1}{2}c_i\sin(z) + z^2 \sum_{j=1}^{i-1} a_{ij}\sin(c_j z) = 0.$$

Where $z = \alpha h$.

2.3. Derivation of the proposed scheme

The proposed scheme is based on the "Three-step third order hybrid method" presented in [11]:

Table 3. Coefficients of ThHM3

-2	0		
0	0	0	
-3	$\frac{5}{4}$	$\frac{1}{4}$	0
	$\frac{3}{8}$	$\frac{29}{24}$	$-\frac{1}{12}$

Obviously, $s = 3$ from Table (3). Now, substitute same in the trig. conditions while holding all the internal coefficients (c_i and a_{ij}) constant, we obtain

$$\cos(z) = \frac{3}{2} - \frac{1}{2}\cos(2z) - z^2(b_1\cos(2z) + b_2 + b_3\cos(3z)),$$
$$\sin(z) = \frac{1}{2}\sin(2z) - z^2(-b_1\sin(2z) - b_3\sin(3z)).$$

That is a system of two equations in three unknown parameters, implying one degree of freedom. The one free parameter could be taken from Table (3) above, but we don't want any of the update stage coefficients to be constant. Hence, we choose one additional equation to augment the number of equations to be solved. The variable coefficients are obtained as follows:

$$b_1 = -\frac{3}{4}\,\frac{\sin(3z)z^2 + 12\sin(z)\cos(z) - 12\sin(z)}{z^2(9\sin(2z) - 4\sin(3z))},$$
$$b_2 = \frac{1}{4}\,\frac{N_1}{z^2(9\sin(2z) - 4\sin(3z))},$$
$$b_3 = \frac{1}{4}\,\frac{3\sin(2z)z^2 + 16\sin(z)\cos(z) - 16\sin(z)}{z^2(9\sin(2z) - 4\sin(3z))},$$

where

$$N_1 = -3\sin(2z)\cos(3z)z^2 + 3\cos(2z)z^2\sin(3z) + 36\sin(z)\cos(z)\cos(2z) - 16\sin(z)\cos(z)\cos(3z) - 36\sin(z)\cos(2z) + 16\sin(z)\cos(3z) - 36\sin(2z)\cos(z) + 16\cos(z)\sin(3z) - 18\cos(2z)\sin(2z) + 8\cos(2z)\sin(3z) + 54\sin(2z) - 24\sin(3z).$$

But observe that as $z \to 0$ there would be heavy cancellations. So, Taylor expansion of the coefficients would be used. The corresponding values after the expansion are:

$$b_1 = \frac{3}{8} + \frac{39\,z^4}{320} - \frac{2627\,z^6}{16128} + O\left(z^8\right),$$
$$b_2 = \frac{29}{24} + \frac{3\,z^4}{320} + \frac{26309\,z^6}{725760} + O\left(z^8\right),$$
$$b_3 = -\frac{1}{12} - \frac{13\,z^4}{240} + \frac{2627\,z^6}{36288} + O\left(z^8\right).$$

2.4. *Confirmation of order of convergence*

The order of the scheme can be confirmed by substituting the coefficients back to algebraic order conditions to check the conditions that are recovered.

$$\sum b_i = \frac{3}{2} + \frac{37\,z^4}{480} - \frac{243\,z^6}{4480} + O\left(z^8\right)$$
$$\sum b_i c_i = -\frac{1}{2} - \frac{13\,z^4}{160} + \frac{2627\,z^6}{24192} + O\left(z^8\right)$$
$$\sum b_i c_i^2 = \frac{3}{4} + O\left(z^{14}\right)$$
$$\sum b_i a_{i,j} = -\frac{1}{8} - \frac{13\,z^4}{160} + \frac{2627\,z^6}{24192} + O\left(z^8\right).$$

It can be seen that the order conditions are recovered as z approaches zero. Hence, by the order of convergence stated in [11], the scheme is of order three.

3. Numerical Results

In this section, the proposed scheme is validated by solving a few examples of problems with known exact solutions. The problems are:

Problem 1 (Inhomogeneous Problem)

$$\frac{d^2y(x)}{dx^2} = -y(x) + x, \quad y(0) = 1, \quad y'(0) = 2.$$

Exact solution: $y(x) = \sin(x) + \cos(x) + x$.

Source: [1,11,12].$x \in [0, 100]$

Problem 2 (Duffing Problem)

$$y'' + y + y^3 = F\cos(vx), \quad y(0) = 0.200426728067,$$
$$y'(0) = 0. \text{ where } \quad F = 0.002 \quad \text{and} \quad v = 1.01.$$

Exact solution: $y(x) = \sum_{i=0}^{4} v_{2i+1} \cos[(2i+1)vx]$,

where $v_1 = 0.200179477536$, $v_3 = 0.246946143 \times 10^{-3}$,
$v_5 = 0.304014 \times 10^{-6}$, $v_7 = 0.374 \times 10^{-9}$, and
$v_9 < 10^{-12}$ $\alpha = 1$.

Source: [11,12].$x \in [0, 100]$

Table 4. Maximum Error for Problem 1

h	**TThMH**	**ThHM**
0.125	$1.09000000 \times 10^{-05}$	$9.14000000 \times 10^{-05}$
0.0625	$6.81778300 \times 10^{-07}$	$5.74000000 \times 10^{-06}$
0.03125	$4.27171140 \times 10^{-08}$	$3.59427562 \times 10^{-07}$
0.015625	$2.67374400 \times 10^{-09}$	$2.24843520 \times 10^{-08}$
0.0078125	$1.67950000 \times 10^{-10}$	$1.40043000 \times 10^{-09}$

Table 5. Maximum Error for Problem 2

h	**TThMH**	**ThHM**
0.125	$1.53000000 \times 10^{-06}$	$1.13900000 \times 10^{-05}$
0.0625	$9.93512828 \times 10^{-08}$	$7.19084606 \times 10^{-07}$
0.03125	$6.33294855 \times 10^{-09}$	$4.51587658 \times 10^{-08}$
0.015625	$4.00945820 \times 10^{-10}$	$2.83063643 \times 10^{-09}$
0.0078125	$2.63143000 \times 10^{-11}$	$1.78347304 \times 10^{-10}$

4. Discussion

The proposed scheme is applied on two test problems along sides its base method. The problems are linear non homogeneous and non linear homogeneous, respectively. The methods maintained a remarkable level of accuracy in solving the problems. It is also obvious as h approaches zero the max. error decreases, which indicates convergence. That is to say the fitted scheme converges faster, as its error decrease more than that of the base method, especially on Problem 2.

5. Conclusions

A fitted numerical scheme for numerical integration of oscillatory problems is proposed and derived. The scheme is validated using test problems whose analytical solutions are known. From the results obtained, it can be concluded that the fitted form of improved hybrid method can be more promising in tackling oscillatory problems, especially non linear ones.

Abbreviations

The following abbreviations are used in this manuscript:

ThHM	The three-step two stage improved hybrid method derived in [11]
TThHM	The proposed scheme presented in this paper

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