computer sciences and mathematics forum

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Proceeding Paper

Qutrit– Based Orthogonal Approximations with Inverse–Free Quantum Gate Sets †

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† Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.
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Abstract: The efficient compiling of arbitrary single–qubit gates into a sequence of gates from a finite gate set is of fundamental importance in quantum computation. The exact bounds of this compiling are given by the Solovay–Kitaev theorem, which serves as a powerful tool in compiling quantum algorithms that require many qubits. However, the inverse–closure condition it imposes on the gate set adds a certain complexity to the experimental compilation, making the process less efficient. This was recently resolved by a version of the Solovay–Kitaev theorem for inverse–free gate sets, yielding a significant gain. Considering the recent progress in the field of three–level quantum systems, in which qubits are replaced by qutrits, it is possible to achieve the quantum speedup guaranteed by the Solovay–Kitaev theorem simply from orthogonal gates. Nevertheless, it has not been investigated previously whether the condition of inverse closure can be relaxed for these qutrit–based orthogonal compilations as well. In this work, we answer this positively, by obtaining improved Solovay–Kitaev approximations to an arbitrary orthogonal qutrit gate, to an accuracy ε from a sequence of $O(log^{8.62}(1/\varepsilon))$ orthogonal gates taken from an inverse–free set.

Keywords: quantum computation; quantum gates; applications of group representations to physics

MSC: 81P68; 81P65; 20C35

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Citation: Mahasinghe, A.; Fernando, D.; De Silva, K. Qutrit–Based Orthogonal Approximations with Inverse–Free Quantum Gate Sets. *Comput. Sci. Math. Forum* **2023**, *1*, 0. https://doi.org/

Academic Editor: Firstname Lastname

Published: 28 April 2023

1. Introduction

Qutrit–based quantum computation [1–3] is an emerging trend in the field of quantum computing. In contrast to two–level quantum systems used in standard quantum computing, three–level quantum systems are used in qutrit–based computing. Accordingly, a qubit that has basis states $|0\rangle$ and $|1\rangle$ is replaced by a qutrit having thee basis states $|0\rangle$, $|1\rangle$ and $|2\rangle$. A growing interest in qutrit–based computations can be found from literature, and it has been proven to provide a better platform for quantum communication [4,5].

In standard quantum computation, a qubit undergoes a state change after being operated by a quantum gate[6]. This gate is a unitary matrix of unit determinant (due to normalization factors), boiling down the problem into the action of an element in $SU(2)$, the 2×2 special unitary group. Thus, it would be the group $SU(3)$, the 3×3 special unitary group that defines the action of a quantum gate applied on a qutrit. However, once the physical implementation is considered, there are several advantages in confining to orthogonal gates [7], as the fault-tolerant implementation of the complex phase gate is much more complicated than the orthogonal gates [8]. In this regard, qutrit-based computations with gates in the special orthogonal group $SO(3)$ are of particular importance, and this has been mathematically investigated in detail in [7].

Once the orthogonal gates are concerned, we should know how we can approximate an arbitrary quantum gate by means of a finite sequence of elements taken from that gate set efficiently, as far as both the time and space complexities are concerned. In standard quantum computing, this problem has been answered by Solovay[9] and Kitaev[10], by establishing the remarkable theorem known today as the *Solovay–Kitaev theorem* . According to this theorem, it is possible to approximate an arbitrary 2×2 unitary with unit determinant by a product of $O(\log^4(1/\varepsilon)$ physically realizable 2×2 basic gates to an arbitrary accuracy ε [11,12]. It was shown in [7] that the Solovay–Kitaev theorem applies for qutrit–based orthogonal computations.

Once the applicability of the Solovay–Kitaev theorem to orthogonals in $SO(3)$ is verified, the next concern is the *feasibility*. In this regard, reducing noise [13] and quantum Hamiltonian complexity [14] is of particular importance. These tasks can be more easily accomplished, by lifting the assumption on the essential inclusion of inverses in the Solovay-Kitaev theorem [15]. This remained an open question in standard quantum computation, until it was resolved very recently by Bouland and Giurgica-Tiron [16], with a clever algorithmic procedure for efficiently approximating a unitary gate with an inverse–free gate set.

In this paper, we show that Bouland and Giurgica-Tiron scheme can be modified and applied for orthogonal computations as well, by proving an inverse–free version of the Solovay–Kitaev theorem in $SO(3)$. Accordingly, the remainder of the paper is organised as follows: In Section 2, we present the basic definitions and mathematical tools required. Then we single out the ingredient in the Solovay–Kitaev theorem that requires inverses, and present a proof and an algorithm with an inverse–free gate set for $SO(3)$ in Section 3. Our concluding remarks are found in Section 4.

2. Basic Definitions and Theorems

Definition 1 (Universal quantum gate set)**.** *A set $\mathcal{G}$ of finite single–qubit unitary gates is called A set of universal gates if $\langle \mathcal{G} \rangle$ is dense in $SU(2)$.*

Notation 1. *For $l \geq 0$, $G_l = \{g_1^{\alpha_1} g_2^{\alpha_2} \dots g_l^{\alpha_l} | g_i \in \mathcal{G}, \alpha_i = \pm 1\}$.*

Theorem 1 (Solovay–Kiteav)**.** *Let G be a finite set of elements in $SU(2)$ containing its own inverses, such that $\langle \mathcal{G} \rangle$ is dense in $SU(2)$. Then for any $\varepsilon > 0$, $\mathcal{G}_l$ provides an ε-net for $SU(2)$ where $l = O(log^{3.67}(1/\varepsilon))$.*

The highly constructive proof of this theorem relies on approximations of elements close to the identity. From a rough approximation U_0 to a given element element U, it finds a better approximation to $\Delta = UU_0^\dagger$ by the repeated application of the following lemma, which is the most important ingredient in the proof.

Lemma 1 (Shrinking Lemma)**.** *Let $\mathcal{G}$ be a finite set of elements in $SU(2)$ containing its own inverses such that $\langle \mathcal{G} \rangle$ is dense in $SU(2)$. There exist constants $s, \varepsilon' > 0$ with $s\varepsilon' < 1$ such that, for every $\varepsilon < \varepsilon'$: if $\mathcal{G}_l$ is an ε^2-net for S_ε, then $\mathcal{G}_{5l}$ is a $s\varepsilon^3$-net for $S_{\sqrt{s\varepsilon^3}}$.*

The Solovay-Kitaev algorithm decomposes Δ into a concatenation of four unitary matrices which make the *group commutator* - $VWV^\dagger W^\dagger$. When the gate set is inverse–closed, to approximate an inverse elements as $V^\dagger$, one can reverse the order of the sequence of V-approximation and apply the corresponding inverses. Bouland and Giurgica-Tiron scheme [16] uses $O(\varepsilon^{3/2})$-approximations to inverse elements instead of exact inverses, using the following lemmata. By X, Y, Z we denote pauli operators.

Lemma 2 (Approximating the group commutator [16])**.** *For two unitaries V and W such that $\|V - I\| = O(\varepsilon_{n-1}^{1/2})$ and $\|W - I\| = O(\varepsilon_{n-1}^{1/2})$, assume $\|V - V_{n-1}\| = O(\varepsilon_{n-1})$ and $\|W - W_{n-1}\| = O(\varepsilon_{n-1})$. Additionally assume that $\|V_{n-1}^{\hat{\dagger}} - V_{n-1}^\dagger\| = O(\varepsilon_{n-1}^{3/2})$ and $\|W_{n-1}^{\hat{\dagger}} - W_{n-1}^\dagger\| = O(\varepsilon_{n-1}^{3/2})$. Then, $\|VWV^\dagger W^\dagger - V_{n-1} W_{n-1} V_{n-1}^{\hat{\dagger}} W_{n-1}^{\hat{\dagger}}\| = O(\varepsilon_{n-1}^{3/2})$.*

Lemma 3 ([16])**.** *If $\|X - X^{'}\| \leq \varepsilon$, then the error $X - X^{'}$ is suppressed in the direction of X. Specifically there are complex scalars x_0, x_2, x_3 of order $O(\varepsilon)$ such that: $X^{'} = X + x_0 I + x_2 Y + x_3 Z + O(\varepsilon^2)$.*

Lemma 4 ([16])**.** *If $\|X - X^{'}\| \leq \varepsilon$ and $\|Y - Y^{'}\| \leq \varepsilon$ then the construction $J_2(X^{'}, Y^{'}) \equiv X^{'}Y^{'}X^{'}Y^{'2}X^{'}Y^{'}X^{'}$ is an ε^2-approximation to the identity, i.e. $\|J_2(X^{'}, Y^{'}) - I\| = O(\varepsilon^2)$.*

Lemma 5 ([16])**.** *Assume that $\|X - X^{'}\| \leq \varepsilon, \|Y - Y^{'}\| \leq \varepsilon, \|Z - Z^{'}\| \leq \varepsilon$. Also assume that given a unitary V, we have an ε-approximate inverse $\hat{V}^{\dagger}$ such that $\|\hat{V}^{\dagger} - V^{\dagger}\| \leq \varepsilon$. Then the construction $\hat{\hat{V}}^{\dagger} = X^{'}(\hat{V}^{\dagger}V)Y^{'}X^{'}(\hat{V}^{\dagger}V)Y^{'2}X^{'}(\hat{V}^{\dagger}V)Y^{'}X^{'}\hat{V}^{\dagger}$ is an ε^2-approximation to $V^{\dagger}$, i.e. $\|\hat{\hat{V}}^{\dagger}V - I\| = O(\varepsilon^2)$.*

Since $\hat{\hat{V}}^{\dagger}$ is an $O(\varepsilon^2)$-approximation to $V^{\dagger}$ we can use it for the execution of the Solovay–Kitaev algorithm, as it has yielded a tighter precision than Lemma 2. Accordingly, we focus on presenting a modified version of the shrinking lemma, which is the only portion of the Solovay–Kitaev theorem that makes use of inverses. Its proof we wish to present in Section 3 requires the following map.

Cornwell's map The two–to–one mapping known as the *Cornwell's map* is a homomorphism from $SU(2)$ to $SO(3)$. It maps the element $U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$ to

$$\rho(U) = \begin{pmatrix} Re(\alpha^2 - \beta^2) & Im(\alpha^2 - \beta^2) & -2Re(\alpha\beta) \\ Im(\alpha^2 - \beta^2) & Re(\alpha^2 + \beta^2) & 2Im(\alpha\beta) \\ 2Re(\alpha\bar{\beta}) & 2Im(\alpha\bar{\beta}) & |\alpha|^2 - |\beta|^2 \end{pmatrix}. \quad (1)$$

Lemma 6 ([7])**.** *For any two $U, V \in SU(2)$, if $\|U - V\| < \varepsilon$, then $\|\rho(U) - \rho(V)\| < O(\varepsilon)$.*

3. Results

3.1. Universal Sets in $SU(2)$ and $SO(3)$

Lemma 7. *If $\mathcal{G}$ is a universal set in $SO(3)$, then set: $\mathcal{K} = \{U \in SU(2); \rho(U) \in \mathcal{G}\}$ is universal in SU(2).*

Proof. Let $\varepsilon > 0$. Let $\mathcal{G}$ be universal in $SO(3)$. Then $\mathcal{G}$ is finite and $\langle\mathcal{G}\rangle$ is dense in $SO(3)$. Hence, $\forall P \in SO(3), \exists Q \in \langle\mathcal{G}\rangle : \|P - Q\| < \varepsilon$. Then $\mathcal{K} = \{U \in SU(2); \rho(U) \in \mathcal{G}\}$ is finite. Let $U \in SU(2)$ then $\rho(U) \in SO(3)$. $\exists Q \in \langle\mathcal{G}\rangle$ such that,

$$\|\rho(U) - Q\| = \|\rho(U) - \rho(V)\| < \varepsilon, \quad (2)$$

where $Q = \rho(V)$ for some $V \in \langle\mathcal{K}\rangle$. By (1) we obtain the results, $\|U - I\|^2 = 2|1 - \alpha|^2 + 2|\beta|^2$ and $\|\rho(U) - \rho(I)\|^2 = |1 - \alpha^2 + \beta^2|^2 + |1 - \alpha^2 - \beta^2|^2 + 4|\alpha\beta|^2 + 4|\alpha\bar{\beta}|^2 + (1 - |\alpha|^2 - |\beta|)^2$. We observe $\|\rho(U) - \rho(I)\|^2 = (1 - |\alpha|^2) + |\alpha|^4 + 8|\alpha||\beta| + \|U - I\|^2$, which implies $\|\rho(U) - \rho(I)\| \geq \|U - I\|$. Forthermore, unitary invariance implies that,

$$\|\rho(U) - \rho(V)\| \geq \|U - V\|. \quad (3)$$

From (2) and (3), $\|U - V\| \leq \|\rho(U) - \rho(V)\| < \varepsilon$, from which we can conclude that $\langle\mathcal{K}\rangle$ is dense in $SU(2)$. Therefore, $\mathcal{K}$ makes a universal set in $SU(2)$. □

3.2. Modified Shrinking Lemma

Lemma 8. *Let $\mathcal{G}$ be a finite set of elements in $SU(2)$ such that $\langle\mathcal{G}\rangle$ is dense in $SU(2)$. There exists constants $s, \varepsilon^{'}$ with $s\varepsilon^{'} < 1$ such that for every $\varepsilon \leq \varepsilon^{'}$: If $\mathcal{G}_l$ is an ε^2-net for S_{ε} then $\mathcal{G}_{33l}$ is an $s\varepsilon^3$-net for $S_{\sqrt{s\varepsilon^3}}$.*

Proof. Assume $\mathcal{G}_l$ is an ε^2-net for S_ε for some $\varepsilon > 0$. Following the steps of the standard proof, we first prove that, $(\mathcal{G}_l, \varepsilon^2, S_\varepsilon) \rightarrow (\mathcal{G}_{32l}, s\varepsilon^3, S_{\varepsilon^2})$.

Let $U \in S_{\varepsilon^2}$. Since U is a special unitary matrix, we can find $\vec{x} \in \mathbb{R}^3$ such that $U = u(\vec{x}) = \exp(-\frac{i}{2}\vec{x}.\vec{\sigma})$. Since $U \in S_{\varepsilon^2}$, $\|U - I\| < \varepsilon^2$. Following some computational steps we can obtain, $\|U - I\| = 4\sin\frac{|\vec{x}|}{4}$. Since $\|U - I\| < \varepsilon^2$, $4\sin\frac{|\vec{x}|}{4} < \varepsilon^2$, and using the Taylor expansion we get, $|\vec{x}| < \varepsilon^2 + O(\varepsilon^6)$. Choose $\vec{y}, \vec{z} \in \mathbb{R}^3$ such that $\vec{x} = \vec{y} \times \vec{z}$ and $|\vec{y}|, |\vec{z}| < \varepsilon$ such that $|\vec{x}| = |\vec{y} \times \vec{z}| \leq |\vec{y}||\vec{z}|$. With $|\vec{y}|, |\vec{z}| < \varepsilon$ we ensure that $u(\vec{y}), u(\vec{z}) \in S_\varepsilon$. Since $\mathcal{G}_l$ is an ε^2-net for S_ε we can find $U_{y_0}, U_{z_0} \in \mathcal{G}_l \cap S_\varepsilon$ such that, $\|U_{y_0} - u(\vec{y})\|, \|U_{z_0} - u(\vec{z})\| \leq \varepsilon^2$. Since $U_{y_0}, U_{z_0} \in S_\varepsilon$, we can find $\vec{y_0}, \vec{z_0} \in \mathbb{R}^3$ such that $|\vec{y_0}|, |\vec{z_0}| < \varepsilon + O(\varepsilon^3)$ where $U_{y_0} = u(\vec{y_0}) = \exp(-\frac{i}{2}\vec{y_0}.\vec{\sigma})$, $U_{z_0} = u(\vec{z_0}) = \exp(-\frac{i}{2}\vec{z_0}.\vec{\sigma})$. Consequently, $\|u(\vec{y_0}) - I\|$, $\|u(\vec{z_0}) - I\| < \varepsilon$. Since $\|u(\vec{y_0}) - I\| < \varepsilon$, from unitary invariance we get $\|u(\vec{y_0})^\dagger - I\| < \varepsilon$. Hence, $u(\vec{y_0})^\dagger$, $u(\vec{z_0})^\dagger \in S_\varepsilon$. Following a similar approach, we can find $\vec{y_1}, \vec{z_1} \in \mathbb{R}^3$ such that $\|u(\vec{y_0})^\dagger - u(\vec{y_1})\|, \|u(\vec{z_0})^\dagger - u(\vec{z_1})\| \leq \varepsilon^2$ where $u(\vec{y_1}) = \exp(-\frac{i}{2}\vec{y_1}.\vec{\sigma})$, $u(\vec{z_1}) = \exp(-\frac{i}{2}\vec{z_1}.\vec{\sigma}) \in S_\varepsilon$ and thus $|\vec{y_1}|, |\vec{z_1}| < \varepsilon + O(\varepsilon^3)$. Recall from Lemma 3, $\|X' - X\| < \varepsilon$ and hence $X'X^\dagger \in S_\varepsilon$. Therefore we can approximate $X'X^\dagger$ from elements of $\mathcal{G}_l$ such that there exists $\vec{q} \in \mathbb{R}^3$ where $X' = Xu(\vec{q}) = X\exp(-\frac{i}{2}\vec{q}.\vec{\sigma})$. Similarly, $Y' = Yu(\vec{p}) = Y\exp(-\frac{i}{2}\vec{p}.\vec{\sigma})$. Note that we can restate Pauli matrices in the form $X = i\exp(-i\frac{\pi}{2}X)$ and $Y = i\exp(-i\frac{\pi}{2}Y)$.

Let,

$$\begin{aligned} U_V = & Xu(\vec{q})u(\vec{y_1})u(\vec{y_0})Yu(\vec{p})Xu(\vec{q})u(\vec{y_1})u(\vec{y_0})(Yu(\vec{p}))^2Xu(\vec{q}) \\ & u(\vec{y_1})u(\vec{y_0})Yu(\vec{p})Xu(\vec{q})u(\vec{y_1}), \end{aligned}$$

$$\begin{aligned} U_W = & Xu(\vec{q})u(\vec{z_1})u(\vec{z_0})Yu(\vec{p})Xu(\vec{q})u(\vec{z_1})u(\vec{z_0})(Yu(\vec{p}))^2Xu(\vec{q}) \\ & u(\vec{z_1})u(\vec{z_0})Yu(\vec{p})Xu(\vec{q})u(\vec{z_1}). \end{aligned}$$

Then from the Taylor expansion of the terms in U_V and U_W we obtain, $U_Vu(\vec{y_0}) = I + O(\varepsilon^4)$ and $U_Wu(\vec{z_0}) = I + O(\varepsilon^4)$. Now we prove that $U = u(\vec{x})$ in S_{ε^2} can be approximated by elements in S_ε.

$$\begin{aligned} \|u(\vec{x}) - u(\vec{y_0})u(\vec{z_0})U_VU_W\| \leq & \|u(\vec{x}) - u(\vec{y_0})u(\vec{z_0})u(\vec{y_0})^\dagger u(\vec{z_0})^\dagger\| \\ & + \|u(\vec{y_0})u(\vec{z_0})u(\vec{y_0})^\dagger u(\vec{z_0})^\dagger - u(\vec{y_0})u(\vec{z_0})U_VU_W\| \end{aligned} \quad (4)$$

From the proof of the standard shrinking lemma,

$$\|u(\vec{x}) - u(\vec{y_0})u(\vec{z_0})u(\vec{y_0})^\dagger u(\vec{z_0})^\dagger\| \leq O(\varepsilon^3). \quad (5)$$

Taking the second term of (4) as D_2,

$$\begin{aligned} D_2 &= \|u(\vec{y_0})u(\vec{z_0})u(\vec{y_0})^\dagger u(\vec{z_0})^\dagger - u(\vec{y_0})u(\vec{y_0})U_VU_W\| \\ &= \|u(\vec{y_0})u(\vec{z_0})u(\vec{y_0})^\dagger(u(\vec{z_0})^\dagger - U_W) + u(\vec{y_0})u(\vec{z_0})(u(\vec{y_0})^\dagger - U_V)u(\vec{y_0})^\dagger \\ &\quad - u(\vec{y_0})u(\vec{z_0})(u(\vec{y_0})^\dagger - U_V)(u(\vec{z_0})^\dagger - U_W)\| \\ &\leq O(\varepsilon^3) \end{aligned} \quad (6)$$

From (5) and (6) we obtain the desired result, $\|u(\vec{x}) - u(\vec{y_0})u(\vec{z_0})U_VU_W\| \leq O(\varepsilon^3)$.

Now we move to the latter part of the proof: $(\mathcal{G}_{32l}, s\varepsilon^3, S_{\varepsilon^2}) \rightarrow (\mathcal{G}_{33l}, s\varepsilon^3, S_{\sqrt{s\varepsilon^3}})$. Let $U \in S_{\sqrt{s\varepsilon^3}}$. Since $U \in S_{\sqrt{s\varepsilon^3}}$, $U \in S_\varepsilon$. By the initial assumption we know $\mathcal{G}_l$ is an ε^2-net for S_ε. Then there exist $U_I \in \mathcal{G}_l$ such that $\|U - U_I\| \leq \varepsilon^2$, which implies $UU_I^\dagger \in S_{\varepsilon^2}$. Now since $\mathcal{G}_{32l}$ is a $s\varepsilon^3$-net for S_{ε^2}, from above proof we can find $\vec{y_0}, \vec{z_0}, \vec{y_1}, \vec{z_1}, \vec{p}, \vec{q} \in \mathbb{R}^3$ such that $\|UU_I^\dagger - u(\vec{y_0})u(\vec{z_0})U_{(VW)}\| < s\varepsilon^3$, where $U_{(VW)} = Xu(\vec{q})u(\vec{y_1})u(\vec{y_0})Yu(\vec{p})Xu(\vec{q})u(\vec{y_1})u(\vec{y_0})(Yu(\vec{p}))^2$ $Xu(\vec{q})u(\vec{y_1})u(\vec{y_0})Yu(\vec{p})Xu(\vec{q})u(\vec{y_1})Xu(\vec{q})u(\vec{z_1})u(\vec{z_0})Yu(\vec{p})Xu(\vec{q})u(\vec{z_1})u(\vec{z_0})(Yu(\vec{p}))^2Xu(\vec{q})$ $u(\vec{z_1})u(\vec{z_0})Yu(\vec{p})$ $Xu(\vec{q})u(\vec{z_1})$, thus U is approximated by a sequence of $33l$ elements from $\mathcal{G}$. □

3.3. Inverse–Free Orthogonal Approximations

Theorem 2 (Inverse-free Solovay–Kitaev theorem in $SO(3)$)**.** *Let $\mathcal{G}$ be a universal set in $SO(3)$. Then for any $\varepsilon > 0$, $\mathcal{G}_l$ provides an ε-net for $SO(3)$ where $l = O(log^{\gamma}(1/\varepsilon))$ and* $\gamma \simeq 8.62$.

Proof. Let $\varepsilon > 0$ and $U \in SO(3)$. The surjectivity of the Cornwell's map guarantees the existence of $V \in SU(2)$ such that $\rho(V) = U$. Since $\mathcal{G}$ is a universal set in $SO(3)$, from Lemma 7 we know that $\mathcal{K} = \{U \in SU(2); \rho(U) \in \mathcal{G}\}$ is universal in $SU(2)$. Since the Solovay–Kiteav theorem holds in $SU(2)$, we can obtain $V_1, V_2, \ldots, V_l \in \mathcal{K}$ such that, $\|V - V_1V_2 \ldots V_l\| < \varepsilon$. From Lemma 6, $\|\rho(V) - \rho(V_1V_2 \ldots V_l)\| < O(\varepsilon)$. Also, the homomorphism ρ readily gives $\rho(V_1V_2 \ldots V_l) = \rho(V_1)\rho(V_2) \ldots \rho(V_l)$. Then $\|U - \rho(V_1)\rho(V_2) \ldots \rho(V_l)\| < O(\varepsilon)$. Since $V_1, V_2, \ldots, V_l \in \mathcal{K}$, we have $\rho(V_1), \rho(V_2), \ldots, \rho(V_l) \in \mathcal{G}$. Then we can conclude that U can be approximated by $\mathcal{G}_l$ where $l = O(log^{\gamma}(1/\varepsilon)$ and $\gamma \simeq 8.62$. □

Now we provide an algorithmic procedure of approximating orthogonal matrices in $SO(3)$ with the use of a universal gate set of orthogonal matrices. The steps closely follow the ones in the algorithms in [7,16]. For detailed descriptions of the steps, the readers are encouraged to refer to those algorithms.

Algorithm 1 Inverse-free Solovay–Kitaev algorithm in $SO(3)$

Require: Any universal gate set $G \subset SO(3)$
function Solovay–Kitaev(S, n)
set U = SO3ToSU2(S)
if $n == 0$ **then**
  return basic approximation U
else
  set U_{n-1} = Solovay–Kitaev$(U, n-1)$
  set V, W = GC-Decompose$(UU_{n-1}^{\dagger})$
  set V_{n-1} = Solovay–Kitaev$(V, n-1)$
  set W_{n-1} = Solovay–Kitaev$(W, n-1)$
  set $V_{n-1}^{\hat{\dagger}}$ = Solovay–Kitaev$(V_{n-1}^{\dagger}, n-1)$
  set $W_{n-1}^{\hat{\dagger}}$ = Solovay–Kitaev$(W_{n-1}^{\dagger}, n-1)$
  set X_{n-1} = Solovay–Kitaev$(X, n-1)$
  set Y_{n-1} = Solovay–Kitaev$(Y, n-1)$
  set $V_{n-1}^{\hat{\hat{\dagger}}} = X_{n-1}V_{n-1}^{\hat{\dagger}}V_{n-1}Y_{n-1}X_{n-1}V_{n-1}^{\hat{\dagger}}V_{n-1}(Y_{n-1})^2X_{n-1}V_{n-1}^{\hat{\dagger}}V_{n-1}Y_{n-1}X_{n-1}V_{n-1}^{\hat{\dagger}}$
  set $W_{n-1}^{\hat{\hat{\dagger}}} = X_{n-1}W_{n-1}^{\hat{\dagger}}W_{n-1}Y_{n-1}X_{n-1}W_{n-1}^{\hat{\dagger}}W_{n-1}(Y_{n-1})^2X_{n-1}W_{n-1}^{\hat{\dagger}}W_{n-1}Y_{n-1}X_{n-1}W_{n-1}^{\hat{\dagger}}$
  set $U_n = V_{n-1}W_{n-1}V_{n-1}^{\hat{\hat{\dagger}}}W_{n-1}^{\hat{\hat{\dagger}}}U_{n-1}$
  set $\tilde{S}$ = SU2ToSO3(U_n)
  return $\tilde{S}$
end if

4. Concluding Remarks

The purpose of the study was to check the possibility of efficient and noiseless quantum computation with three–level quantum systems, for which an inverse–free version of the Solovay–Kitaev theorem could be of immense help. Following the previous work [7], we considered orthogonal gates instead of unitary gates. Accordingly, we used the inverse–free version of the Solovay–Kitaev theorem developed by Bouland and Giurgica-Tiron [16] and verified that a version of it is applicable to $SO(3)$. We concluded that efficient orthogonal compiling is possible with a finite universal gate set which holds the mere condition of densely generating the orthogonal space $SO(3)$. More specifically, we obtained $O(log^{8.62}(1/\varepsilon))$ and $O(log^{3.97}(1/\varepsilon))$ as the values for sequence length and run time for one such approximation of an orthogonal matrix in $SO(3)$ confirming Solovay-Kitaev executions are possible for orthogonals in $SO(3)$ with a universal gate set.

It is evident that we require a greater value for the exponent in the poly–logarithmic asymptotic of the algorithm than the one in the standard algorithm. The inverse construction approach we utilized yielded approximations to $O(\varepsilon^2)$ precision, whereas it only requires approximations to $O(\varepsilon^{3/2})$ precision. It would be an interesting future task to investigate the possibility of lowering this value of the exponent in the poly–logarithmic asymptotic by producing approximations to the $O(\varepsilon^{3/2})$ precision.

Author Contributions: Conceptualization and Methodology: A.M.; Formal Analysis and Original Draft Preparation: D.F.; Review and Validation: K.D.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement:

Conflicts of Interest: The authors declare no conflict of interest.

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