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# Estimating the Dependence Parameter of Farlie-Gumbel-Morgenstern Type Bivariate Gamma Distribution Using Ranked Set Sampling †

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**Abstract:** The goal of the present work is to estimate the nonlinear correlation between two random variables when the sample is drawn from a Farlie-Gumbel-Morgenstern (FGM) type bivariate gamma distribution. In the context of estimating dependence parameter, maximum likelihood (ML) methodology is used. Thus, the present work offers ML estimators based on simple random sampling (SRS) and ranked set sampling (RSS). Also, we consider generalized modified RSS (GMRSS), which only requires a single rank to obtain a sample. Using GMRSS, we aim to observe the effect of $r$th order statistic and its concomitant on the ML estimator. According to the Monte Carlo simulation, it is clearly seen that RSS provides an ML estimator as efficient as the ML estimator based on SRS. On the other hand, it is appeared that the ML estimator based on GMRSS (with minimum or maximum ranked pairs) is the best option among the studied ML estimators. Moreover, these findings are made even more meaningful by the fact that GMRSS is easier to obtain than SRS and RSS.

**Keywords:** ranked set sampling; algorithm for sampling data; order statistics; dependence parameter; FGM family; maximum likelihood estimation

**MSC:** 62D99; 62F10; 62G30; 62H05; 65C05



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## 1. Introduction

Farlie-Gumbel-Morgenstern (FGM) is a quite attractive distribution family since its simple form and provide a modelling the dependence between two random variables. The family was introduced by Morgenstern [1], Gumbel [2] and Farlie [3]. Several researchers have studied on FGM distribution family, see [4–10]. D'Este [11] and Gupta & Wong [12] discussed bivariate gamma distribution, which is derived from FGM family. The joint cumulative distribution function (CDF) of the random variables $X$ and $Y$ on $\mathbb{R}^2$ is given by

$$H(x,y) = F(x)G(y)[1 + \lambda(1 - F(x))(1 - G(y))] \tag{1}$$

with the corresponding joint probability density function (PDF)

$$h(x,y) = f(x)g(y)[1 + \lambda(1 - 2F(x))(1 - 2G(y))] \tag{2}$$

where $f(x)$ and $g(y)$ are gamma PDFs,

$$f(x) = \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1}, \qquad x > 0, \qquad \alpha \geq 0, \tag{3}$$

$$g(y) = \frac{1}{\Gamma(\beta)} e^{-y} y^{\beta-1}, \qquad y > 0, \qquad \beta \geq 0, \tag{4}$$

with their CDFs $F(x)$ and $G(y)$,

$$F(x) = \frac{\gamma(\alpha, x)}{\Gamma(\alpha)}, \qquad G(y) = \frac{\gamma(\beta, y)}{\Gamma(\beta)} \tag{5}$$

Also, $\lambda$ is a dependence parameter with $-1 \leq \lambda \leq 1$. The random variables are said to be independent if the dependence parameter equals to zero. In Equations (3)–(5), $\Gamma(.)$ and $\gamma(.,.)$ are gamma and incomplete gamma functions, respectively. For the characteristic properties of FGM type bivariate gamma distribution see Kotz et al. [13].

Ranked set sampling (RSS) was introduced by McIntyre [14] as a cost-effective sampling method. The estimation of the dependence parameter using RSS and its modifications has been addressed by some authors, such as Stokes [15], Modarres & Zheng [16], Al-Saleh & Samawi [17], Sevil & Yildiz [18].

Let us introduce the RSS procedure ($k$: set size and $m$: number of cycle) as follows,

Step I: Select $k^2$ pairs from the population for the $j$th cycle.
Step II: Divide the pairs into the $k$ sets at random.
Step III: Select $r$th order statistics and its concomitant from the $r$th set, where $r = 1, 2, \cdots, k$.
Step IV: Steps I-III are repeated $m$ cycle, $j = 1, 2, \cdots, m$.

**Example 1.** *Let $k = 3$ and $m = 1$. A sample of size $k^2 = 9$ is selected from the population and the rth set is denoted as $Set_r = \{(X_{r1}, Y_{r1}), (X_{r2}, Y_{r2}), (X_{r3}, Y_{r3})\}$, where $r = 1, 2, 3$. In each set, the Xs are ranked from the smallest to the largest. Additionally, Ys ranked according to the variable X. From the rth set, the rth order statistics and its concomitant are selected. Then, RSS with one cycle is represented by $\left\{\left(X_{(1)1}, Y_{[1]1}\right), \left(X_{(2)1}, Y_{[2]1}\right), \left(X_{(3)1}, Y_{[3]1}\right)\right\}$. If $m \neq 1$, then RSS is $\left(X_{(r)j}, Y_{[r]j}\right)$ where $j = 1, 2, \cdots, m$.*

Other ranked based sampling design is called generalized modified RSS (GMRSS) which is defined by Sevil and Yildiz [18]. Only the $r$th order statistics $X_{(r)j}$ and its concomitants $Y_{[r]j}$ are selected from all sets in the GMRSS procedure. Thus, the sampling design is called GMRSS($R = r$) and is denoted by $\left(X_{\kappa(r)j}, Y_{\kappa[r]j}\right)$ where $\kappa = 1, 2, \cdots, k$, $j = 1, 2, \cdots, m$ and $r \in [1, k]$.

To the authors' best knowledge, there has been limited study on estimating the dependence parameter especially when the random variables are nonlinearly related. Moreover, the fact that FGM is a useful distribution family makes this study more meaningful. Rest of the paper organized as follows. First, we provide some algorithms to generate sampling data from FGM type bivariate gamma distribution in Section 2. Then, in Section 3, we examine ML estimators based on SRS, RSS and GMRSS($R = r$). Section 4 provides biases and efficiencies of the ML estimators based on SRS, RSS and GMRSS. Finally, we give some critical comments for the simulation results in Section 5.

## 2. Algorithms to Generate Samples

In this section, we focus on using the copula as a tool. Supposed that $U = F(x)$ and $V = G(y)$ where both $X$ and $Y$ are continuous. Thus, the joint distribution of $U$ and $V$ is called copula that is represented by $C$ with domain $[0, 1]^2$. According to Sklar's theorem [19], we define a bivariate copula $C$

$$H(x, y) = P[F(X) \leq F(x), G(Y) \leq G(y)] = C_\lambda(F(x), G(y)) = C_\lambda(u, v) \tag{6}$$

where $H(x, y)$ is joint CDF on $\mathbb{R}^2$ with marginal CDFs $F(x)$ and $G(y)$. If $F(x)$ and $G(y)$ are continuous, $C_\lambda(u, v)$ is unique. Let $F^{-1}(u) = \{x : F(x) \geq u\}$ be inverse function of $F(x)$, then $C_\lambda(u,v) = H\big(F^{-1}(u), G^{-1}(v)\big)$ for $(u,v) \in [0,1]^2$. By taking partial derivatives of $C_\lambda(u,v)$ with respect to $u$ and $v$, the density function of the bivariate copula $c_\lambda(u,v)$ can be obtained,

$$c_\lambda(u,v) = \frac{\partial^2 C_\lambda(u,v)}{\partial u \partial v}, \tag{7}$$

where $\lambda$ $(-1 \leq \lambda \leq 1)$ is the dependence parameter of the bivariate copula. Thus, the form of density function is $h(x,y) = c_\lambda(u,v) f(x) g(y)$

Now, the copula form of FGM is $C_\lambda(u,v) = uv[1 + \lambda(1-u)(1-v)]$ where $u, v \in [0,1]$. Thanks to the Sklar's theorem [19], we only need generate such of a pair $(u,v)$ on $[0,1]^2$ whose joint distribution function is $C_\lambda(u,v)$. Algorithm for generating a pair $(u,v)$ of observations of uniform $(0,1)$ random variables $(U,V)$ was introduced by Johnson [20] and Nelsen [21]. This algorithm uses conditional distribution approach. Assumed that $c_u(v)$ is conditional distribution function for $V$ given $U = u$.

$$c_u(v) = P[V \leq v | U = u] = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u,v)}{\Delta u} = \frac{\partial C(u,v)}{\partial u} = [1 + \lambda(1-2u)]v - \lambda(1-2u)v^2 \tag{8}$$

The following theorem demonstrates that $c_u(v)$ exists and is nondecreasing almost everywhere $v \in [0,1]$.

**Theorem 1** (Nelsen [21])**.** *Let C be a copula. For any $v \in [0,1]$, the partial derivative $\partial C(u,v)/\partial u$ exists and $0 \leq \frac{\partial C(u,v)}{\partial u} \leq 1$ for almost all $u$, and for such $v$ and $u$. Moreover, the function $v \mapsto \partial C(u,v)/\partial u$ is defined and nondecreasing almost everywhere on $v \in [0,1]$.*

Johnson [20] obtained quasi-inverse ($c_u^{-1}$) of $c_u$ by solving the equation, $[1 + \lambda(1-2u)]v - \lambda(1-2u)v^2 = p$, which quadratic in $v$. This equation has one root for $0 < p < 1$ and it is $v = c_u^{-1}(p) = \frac{2p}{b+a}$ where $a = 1 + \lambda(1-2u)$ and $b = \sqrt{a^2 - 4(a-1)p}$. The following algorithm is given for simulating SRS data from FGM type bivariate gamma distribution.

**Algorithm 1** : Generating data from FGM type bivariate gamma distribution.

Step I: Generate $u$ and $p$ from i.i.d. random variates uniform $(0,1)$;
Step II: $a = 1 + \lambda(1-2u)$ and $b = \sqrt{a^2 - 4(a-1)p}$;
Step III: Set $v = c_u^{-1}(p) = \frac{2p}{b+a}$;
Step IV: The desired pair is $(u,v)$;
Step V: $X = F^{-1}(u)$ and $Y = G^{-1}(v)$;
Step VI: RETURN $(X,Y)$.

Where $F^{-1}$ and $G^{-1}$ are inverse functions of $F$ and $G$ which are given by Equation (5). However, there is no closed form of the inverse functions. By using "qgamma" function in R statistical programming language, the desired pair $(x,y)$ can be obtained. To generate a ranked set sample from FGM type bivariate gamma distribution, the Algorithm 2 is proposed in this paper. Using the Algorithm 2, $m$ pairs are generated from each rank $r = 1, 2, \cdots, k$ and RSS is balanced. If only $r$th ranked pairs are selected from all sets, then GMRSS ($R = r$) can be constructed by using the Algorithm 2. Therefore, GMRSS is not balanced. We will see that GMRSS provides more information than SRS and RSS, even if the GMRSS only consists of single ordered pairs.

**Algorithm 2**: Generating RSS data from FGM type bivariate gamma distribution.

Step I: Generate a random sample $(X_i, Y_i)$, $i = 1, 2, \cdots, k^2$, using Algorithm 1 for $j$th cycle;
Step II: Divide the units in the sample randomly into $k$ sets of size $k$ each;
Step III: Rank the units in each set from the smallest to the largest by using the variable $X$;
Step IV: Select the order statistic $X_{(r)j}$ and its concomitant variable $Y_{[r]j}$ from $r$th set;
Step V: RETURN $(X_{(r)j}, Y_{[r]j})$ where $r = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, m$.

The authors will provide these R functions for the Algorithms 1 and 2 upon request.

## 3. Maximum Likelihood Estimates of Dependence Parameter

Let $\Theta = (\alpha, \beta, \lambda)^T$ be a parameter vector where $\alpha$ and $\beta$ are shape parameters of the variables $X$ and $Y$, respectively. Also, $\lambda$ is the dependence parameter. By examining similar studies, it can be seen that there are three different scenarios in which the ML estimator of the dependence parameter is investigated: (i) the parameters of the variables $X$ and $Y$ are known, (ii) the parameters of the variable $X$ (which is more easily accessible) are known, but those of $Y$ are not, and (iii) all parameters are unknown. Beginning with case (i), examination of the ML estimator for the dependence parameter can be meaningful and offer some hints for cases (ii) and (iii).

### 3.1. ML Estimator from Simple Random Sample

Suppose that $(X_i, Y_i)$, $i = 1, \cdots, n$, is a sample drawn at random from $h(x, y)$. Let log-likelihood function is $L_{SRS}(\lambda) = \Sigma_{i=1}^{n} \log(h(x_i, y_i))$. Since the analytically closed expression of $\hat{\lambda}_{SRS}$ is not defined, the ML estimate of $\lambda$ ($\hat{\lambda}_{SRS}$) can be obtained by solving the following equation.

$$\frac{\partial L_{SRS}(\lambda)}{\partial \lambda} = \sum_{i=1}^{n} \frac{h'(x_i, y_i)}{h(x_i, y_i)} = \sum_{i=1}^{n} \frac{(1-2u_i)(1-2v_i)}{1+\lambda(1-2u_i)(1-2v_i)} = 0, \tag{9}$$

where $h'(x_i, y_i)$ is the first derivative of the $h(x_i, y_i)$ with respect to $\lambda$, $u = F'(x)$ and $v = G'(y)$. The variance of $\hat{\lambda}_{SRS}$, $V\left(\hat{\lambda}_{SRS}\right)$ is defined as follows,

$$V\left(\hat{\lambda}_{SRS}\right) = \left\{-E\left[\frac{\partial^2 L_{SRS}(\lambda)}{\partial \lambda^2}\right]\right\}^{-1} = \left\{E\left[\sum_{i=1}^{n}\left(\frac{(1-2u_i)(1-2v_i)}{1+\lambda(1-2u_i)(1-2v_i)}\right)^2\right]\right\}^{-1} \tag{10}$$

### 3.2. ML Estimator from Ranked Set Sample

In this section, the ML estimator based on RSS ($\hat{\lambda}_{RSS}$) is established. It is assumed that $\left(X_{(r)j}, Y_{[r]j}\right)$ is ranked set sample where $r = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, m$. Under the perfect ranking condition, $X_{(r)}$ has the same PDF of $r$th order statistic,

$$f_{r:k}(x) = \frac{k!}{(r-1)!(k-r)!} f(x) F^{(r-1)}(x) (1-F(x))^{k-r}, \tag{11}$$

where $f(x)$ and $F(x)$ are given by Equations (3) and (5), respectively. The joint PDF of $\left(X_{(r)j}, Y_{[r]j}\right)$ can be written as follows,

$$f_{r:k}(x,y) = f_{r:k}(x) f(y|x) = \frac{k!}{(r-1)!(k-r)!} F^{(r-1)}(x)(1-F(x))^{k-r} h(x,y) \tag{12}$$

where $h(x,y)$ is given by Equation (2). Thus, log-likelihood function can be expressed as $L_{RSS}(\lambda) = \Sigma_{j=1}^{m}\Sigma_{r=1}^{k}\log(f_{r:k}(x_{(r)j}, y_{[r]j}))$. Since $\lambda$ appears in $h(x,y)$,

$$\frac{\partial L_{RSS}(\lambda)}{\partial\lambda} = \sum_{j=1}^{m}\sum_{r=1}^{k}\frac{h'(x_{(r)j}, y_{[r]j})}{h(x_{(r)j}, y_{[r]j})} = \sum_{j=1}^{m}\sum_{r=1}^{k}\frac{\left(1-2u_{(r)j}\right)\left(1-2v_{[r]j}\right)}{1+\lambda\left(1-2u_{(r)j}\right)\left(1-2v_{[r]j}\right)} = 0 \tag{13}$$

where $u_{(r)j} = F^{-1}(x_{(r)j})$ and $v_{[r]j} = G^{-1}(y_{[r]j})$. The ML estimator based on RSS ($\hat{\lambda}_{RSS}$) can be found by solving the Equation (15) with a computer algorithm. The variance of $\hat{\lambda}_{RSS}$ is obtained using the following equation.

$$V(\hat{\lambda}_{RSS}) = \left\{-E\left[\frac{\partial^2 L_{RSS}(\lambda)}{\partial\lambda^2}\right]\right\}^{-1} = \left\{E\left[\sum_{j=1}^{m}\sum_{r=1}^{k}\left(\frac{\left(1-2u_{(r)j}\right)\left(1-2v_{[r]j}\right)}{1+\lambda\left(1-2u_{(r)j}\right)\left(1-2v_{[r]j}\right)}\right)^2\right]\right\}^{-1} \tag{14}$$

### 3.3. ML Estimator from Generalized Modified Ranked Set Sample

Let $\left(X_{\kappa(r)j}, Y_{\kappa[r]j}\right)$ is denoted GMRSS($R = r$) where $\kappa = 1,2,\cdots,k$, $j = 1,2,\cdots,m$ and $r \in [1,k]$. The $n = mk$ pairs in this sampling scheme are selected from the $r$th ranked pairs. Assumed that $\hat{\lambda}_{R=r}$ is ML estimator based on GMRSS ($R = r$). Under perfect ranking condition, $X_{(r)}$ has the same PDF of the $r$th order statistic that is given by Equation (13) for each $\kappa$ and $j$. Thus, the joint PDF of $\left(X_{\kappa(r)j}, Y_{\kappa[r]j}\right)$ can be defined as given Equation (16). Log-likelihood function is $L_{RSS}(\lambda) = \Sigma_{j=1}^{m}\Sigma_{\kappa=1}^{k}\log(f_{r:k}(x_{\kappa(r)j}, y_{\kappa[r]j}))$. By taking the first derivative of $L_{GMRSS(R=r)}(\lambda)$ with respect to $\lambda$, the following equation is obtained.

$$\frac{\partial L_{R=r}(\lambda)}{\partial\lambda} = \sum_{j=1}^{m}\sum_{\kappa=1}^{k}\frac{h'(x_{\kappa(r)j}, y_{\kappa[r]j})}{h(x_{\kappa(r)j}, y_{\kappa[r]j})} = \sum_{j=1}^{m}\sum_{\kappa=1}^{k}\frac{\left(1-2u_{\kappa(r)j}\right)\left(1-2v_{\kappa[r]j}\right)}{1+\lambda\left(1-2u_{\kappa(r)j}\right)\left(1-2v_{\kappa[r]j}\right)} = 0 \tag{15}$$

where $u_{\kappa(r)j} = F^{-1}(x_{\kappa(r)j})$ and $v_{\kappa[r]j} = G^{-1}(y_{\kappa[r]j})$. The ML estimator based on GMRSS ($R = r$) is represented by $\hat{\lambda}_{R=r}$ and can be obtained by solving Equation (17). The variance of $\hat{\lambda}_{R=r}$ is described as following,

$$V(\hat{\lambda}_{R=r}) = \left\{-E\left[\frac{\partial^2 L_{R=r}(\lambda)}{\partial\lambda^2}\right]\right\}^{-1} = \left\{E\left[\sum_{j=1}^{m}\sum_{\kappa=1}^{k}\left(\frac{\left(1-2u_{\kappa(r)j}\right)\left(1-2v_{\kappa[r]j}\right)}{1+\lambda\left(1-2u_{\kappa(r)j}\right)\left(1-2v_{\kappa[r]j}\right)}\right)^2\right]\right\}^{-1} \tag{16}$$

## 4. Results

In this section, we present Monte Carlo simulation results. In the simulation, 10,000 samples are generated by using Algorithms 1 and 2. The values of dependence parameters are taken to be $\lambda = \{-0.5, 0, 0.5\}$, the sample sizes to be $n = \{30, 60, 90\}$ and the set sizes to be $k = \{3, 5\}$. In the Tables 1–3, the estimated values $\hat{\lambda}_{\psi}$, relative efficiency (RE) and relative information (RI) are reported, where $\psi$ = SRS, RSS and $R = r$ (GMRSS ($R = r$)). REs of the ML estimators based on RSS and GMRSS ($R = r$) with respect to the ML estimator based on SRS are obtained by using $RE = MSE(\hat{\lambda}_{SRS})/MSE(\hat{\lambda}_{RSS})$ and $RE_{R=r} = MSE(\hat{\lambda}_{SRS})/MSE(\hat{\lambda}_{R=r})$, where $MSE(\hat{\lambda}_{\psi}) = (1/10{,}000)\Sigma_{w=1}^{10{,}000}(\hat{\lambda}_{\psi,w} - \lambda)^2$ for $\psi$=SRS, RSS and $R = r$. On the other hand, the RIs are defined as $RI = FI(\hat{\lambda}_{RSS})/FI(\hat{\lambda}_{SRS})$ and $RI_{R=r} = FI(\hat{\lambda}_{R=r})/FI(\hat{\lambda}_{SRS})$, where FIs are fisher information values $FI(\hat{\lambda}_{\psi}) = -E\left[\partial^2 L_{\psi}(\lambda)/\partial\lambda^2\right]$ for each $\psi$ = SRS, RSS and $R = r$. In the Monte Carlo simulation, the first and second derivatives of the log-likelihoods with respect to $\lambda$ are attained by using a numerical optimization program which is "optim" function in R programming language. In the R function "optim", *method* is chosen as "L-BFGS-B".

**Table 1.** Estimated values ($\hat{\lambda}_{SRS}$, $\hat{\lambda}_{RSS}$), REs and RIs of $\hat{\lambda}_{RSS}$ with respect to $\hat{\lambda}_{SRS}$.

| $\lambda$ | −0.5 | | | | | | 0 | | | | | | 0.5 | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 30 | | 60 | | 90 | | 30 | | 60 | | 90 | | 30 | | 60 | | 90 | |
| $k$ | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 | 3 | 5 |
| $\lambda_{SRS}$ | −0.458 | −0.458 | −0.485 | −0.485 | −0.498 | −0.498 | 0.002 | 0.002 | −0.004 | −0.004 | −0.006 | −0.006 | 0.459 | 0.459 | 0.492 | 0.492 | 0.498 | 0.498 |
| $\lambda_{RSS}$ | −0.455 | −0.449 | −0.49 | −0.489 | −0.498 | −0.497 | −0.001 | 0.000 | −0.001 | 0.000 | 0.001 | 0.003 | 0.458 | 0.456 | 0.482 | 0.486 | 0.493 | 0.496 |
| $RE$ | 0.984 | 0.981 | 1.032 | 1.016 | 0.994 | 1.037 | 1.002 | 1.001 | 1.035 | 1.017 | 1.004 | 1.016 | 0.975 | 0.976 | 0.982 | 0.987 | 0.990 | 0.997 |
| $RI$ | 0.998 | 1.007 | 0.993 | 1.007 | 1.002 | 0.999 | 1.014 | 0.999 | 0.999 | 0.995 | 1.003 | 1.002 | 1.007 | 0.991 | 1.000 | 0.992 | 0.999 | 1.000 |

**Table 2.** Estimated values ($\hat{\lambda}_{R=r}$), REs and RIs of $\hat{\lambda}_{R=r}$ with respect to $\hat{\lambda}_{SRS}$ ($k = 3$).

| $\lambda$ | $n$ | $\hat{\lambda}_{R=1}$ | $\hat{\lambda}_{R=2}$ | $\hat{\lambda}_{R=3}$ | $RE_{R=1}$ | $RI_{R=1}$ | $RE_{R=2}$ | $RI_{R=2}$ | $RE_{R=3}$ | $RI_{R=3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| −0.5 | 30 | −0.461 | −0.417 | −0.471 | 1.139 | 1.222 | 0.695 | 0.569 | 1.130 | 1.229 |
| | 60 | −0.496 | −0.463 | −0.492 | 1.202 | 1.218 | 0.656 | 0.572 | 1.144 | 1.213 |
| | 90 | −0.499 | −0.483 | −0.501 | 1.138 | 1.213 | 0.658 | 0.570 | 1.179 | 1.210 |
| 0 | 30 | 0.005 | −0.001 | −0.004 | 1.175 | 1.216 | 0.707 | 0.583 | 1.154 | 1.207 |
| | 60 | 0.002 | −0.006 | 0.001 | 1.147 | 1.202 | 0.630 | 0.597 | 1.230 | 1.194 |
| | 90 | −0.005 | −0.004 | −0.003 | 1.228 | 1.196 | 0.618 | 0.604 | 1.185 | 1.191 |
| 0.5 | 30 | 0.465 | 0.400 | 0.459 | 1.165 | 1.230 | 0.650 | 0.570 | 1.126 | 1.218 |
| | 60 | 0.492 | 0.467 | 0.488 | 1.177 | 1.216 | 0.660 | 0.574 | 1.193 | 1.223 |
| | 90 | 0.505 | 0.487 | 0.500 | 1.208 | 1.207 | 0.658 | 0.573 | 1.221 | 1.211 |

**Table 3.** Estimated values ($\hat{\lambda}_{R=r}$), REs and RIs of $\hat{\lambda}_{R=r}$ with respect to $\hat{\lambda}_{SRS}$ ($k = 5$).

| $\lambda$ | $n$ | $\hat{\lambda}_{R=1}$ | $\hat{\lambda}_{R=2}$ | $\hat{\lambda}_{R=3}$ | $\hat{\lambda}_{R=4}$ | $\hat{\lambda}_{R=5}$ | $RE_{R=1}$ | $RI_{R=1}$ | $RE_{R=2}$ | $RI_{R=2}$ | $RE_{R=3}$ | $RI_{R=3}$ | $RE_{R=4}$ | $RI_{R=4}$ | $RE_{R=5}$ | $RI_{R=5}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| −0.5 | 30 | −0.489 | −0.432 | −0.379 | −0.426 | −0.475 | 1.456 | 1.667 | 0.792 | 0.689 | 0.543 | 0.399 | 0.753 | 0.686 | 1.460 | 1.665 |
| | 60 | −0.497 | −0.481 | −0.439 | −0.477 | −0.503 | 1.530 | 1.623 | 0.787 | 0.683 | 0.506 | 0.396 | 0.744 | 0.689 | 1.457 | 1.624 |
| | 90 | −0.501 | −0.488 | −0.457 | −0.490 | −0.497 | 1.549 | 1.614 | 0.743 | 0.691 | 0.470 | 0.397 | 0.713 | 0.692 | 1.605 | 1.600 |
| 0 | 30 | −0.005 | 0.002 | −0.002 | 0.002 | −0.005 | 1.424 | 1.571 | 0.792 | 0.695 | 0.602 | 0.406 | 0.801 | 0.703 | 1.451 | 1.581 |
| | 60 | 0.000 | 0.000 | 0.004 | −0.004 | −0.002 | 1.542 | 1.549 | 0.741 | 0.713 | 0.498 | 0.425 | 0.734 | 0.713 | 1.536 | 1.563 |
| | 90 | −0.001 | 0.006 | 0.000 | −0.007 | −0.005 | 1.535 | 1.557 | 0.709 | 0.713 | 0.456 | 0.429 | 0.718 | 0.713 | 1.576 | 1.551 |
| 0.5 | 30 | 0.480 | 0.430 | 0.371 | 0.423 | 0.483 | 1.485 | 1.655 | 0.792 | 0.687 | 0.545 | 0.399 | 0.775 | 0.697 | 1.455 | 1.675 |
| | 60 | 0.496 | 0.476 | 0.439 | 0.475 | 0.496 | 1.497 | 1.630 | 0.775 | 0.684 | 0.523 | 0.398 | 0.724 | 0.690 | 1.503 | 1.610 |
| | 90 | 0.497 | 0.486 | 0.455 | 0.489 | 0.502 | 1.520 | 1.605 | 0.761 | 0.686 | 0.481 | 0.399 | 0.741 | 0.689 | 1.504 | 1.597 |

## 5. Discussion

The present work provides ML estimators based on SRS, RSS and GMRSS ($R = r$) for the dependence parameter of FGM type bivariate gamma distribution. According to the Monte Carlo simulation, we obtain the following findings,

- In Table 1, it is observed that the estimated values $\hat{\lambda}_{SRS}$ and $\hat{\lambda}_{RSS}$ are similar. The estimated values get closer to actual $\lambda$ as sample size increases. Also, when considering the REs and RIs, we can say that the ML estimator based on RSS is as efficient as the ML estimator based on SRS. For researchers who study RSS and its extensions, the result might be a surprise. Similar results, however, were found in Stokes [15] and Sevil and Yildiz [18].
- Table 2 shows that the ML estimators based on GMRSS ($R = 1$) and GMRSS ($R = 3$) has smaller biases than the ML estimator based on GMRSS ($R = 2$). Additionally, it should be noted that compared to SRS, RSS, and GMRSS ($R = 2$), GMRSS ($R = 1$) and GMRSS ($R = 3$) provide more efficient ML estimators. On the other hand, there is no evidence that $RE_{R=r}$ and $RI_{R=r}$ are monotone increasing or decreasing function of the sample size $n$.
- When comparing SRS and RSS, Table 3 demonstrates that GMRSS ($R = 1$) and GMRSS ($R = 5$) have the least biases (aside from $\lambda = 0$). Thus, it is revealed that the ML estimator based on GMRSS ($R = r$) approaches actual $\lambda$ while the set size increases. Also, the highest REs and RIs are obtained when the sample are selected from the minimum ranked ($R = 1$) pairs or maximum ranked ($R = 5$) pairs. According to the Tables 2 and 3, we see that REs and RIs decrease while the rank $r$ is increasing for

$1 \le r \le (k+1)/2$. On the other hand, it is observed that REs and RIs are increasing in $r$ for $(k+1)/2 \le r \le k$.

- Considering that GMRSS design only needs one rank value while RSS requires all rank values, it is obvious that a sample can be obtained by using GMRSS with less effort than RSS. Consequently, the authors recommend using the GMRSS ($R = 1$) or GMRSS ($R = k$) design to estimate the dependence parameter of FGM type bivariate gamma distribution when the set size $k$.

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