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Stochastic Boundary Value Problems via Wiener Chaos Expansion

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Abstract: In this work we study stochastic boundary value problems via a Wiener chaos Expansion in acoustics and linear elasticity. In particular, for both cases we provide the appropriate variational formulation for the stochastic-source Helmholtz equation as well as for the Navier one with stochastic boundary data. The main idea is to reduce our stochastic problems into an infinite hierarchy of deterministic boundary value problems, for each of which an appropriate variational formulation, is considered. Further, we present well-posedness for the above hierarchy of deterministic problems, we give the appropriate linchpin frame with the stochastic problem and we exploit uniqueness and existence arguments for the weighted Wiener chaos solution. Finally, some useful remarks and conclusions are also given.

Keywords: Stochastic boundary value problems, Hermite polynomials, Helmholtz and Navier Equation, Weighted Wiener chaos solution

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1. Introduction

In this paper we study stochastic boundary value problems arising in acoustics and linear elasticity. Our methodology is based upon the use of an appropriate Wiener chaos expansion for the Helmholtz equation with stochastic source, as well as and for the case of the Navier equation with stochastic boundary data. Although the corresponding deterministic problems have been widely studied, there is relatively little work for the corresponding stochastic problems required to incorporate effects of randomness and uncertainty, which turn the problem original partial differential equation (PDE) problem to a stochastic partial differential equation (SPDE) citeKalpineli.

The aim of this work is to establish existence and uniqueness of solutions for stochastic boundary value problems due to Helmholtz and Navier equation. Building on previous work on elliptic and parabolic equations (see e.g. [1–4] and references therein), the key idea is to use the Wiener chaos expansion and decompose the SPDE into an infinite hierarchy of deterministic PDE problems whose properties are well studied, and then compose the solution of the SPDE as a generalized random series, thus allowing us to obtain well posedness results for the SPDE. The results of the present paper are motivated by and can be considered as a first step towards our final goal of applying this method to acoustic and elastic scattering problems for obstacles with various boundary conditions.

Our paper is organized as follows. In Section 2, and for the convenience of the reader, we give preliminaries mathematical notations as well as the appropriate functional space setting. In Section 3, we deal with the stochastic boundary value problem for the Helmholtz equation, for which an analogous approach due to [4,5] is applied.

In Section 4, we study and give results for a stochastic elastic boundary value problem, where the boundary condition is a random variable [2,6]. Finally, in Section 5, we give some useful remarks and conclusions.

2. Mathematical Preliminaries

In this section we present mathematical notations and suitable functional space setting. Initially, we consider the Wiener Chaos Expansion of elements of the space of square-integrable functions defined on the space of tempered distributions [5].

Let $S\left(\mathbb{R}^d\right)$ be the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$, where its dual space $S^*\left(\mathbb{R}^d\right)$ be the space of tempered distributions. We also mention that there exists a unique probability measure P on F, where F is the family of Borel subsets of $S^*\left(\mathbb{R}^d\right)$, such that

$$E\left[e^{i\langle\cdot,\phi\rangle}\right] := \int_{S^*} e^{i\langle\omega,\phi\rangle} dP(\omega) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2\left(\mathbb{R}^d\right)}\right) \forall\phi \in S, \tag{1}$$

where $\langle\omega, \phi\rangle = \omega(\phi)$ is the process of $\omega \in S^*$ on $\phi \in S$ (Bochner-Minlos Theorem), [5].

The Hermite polynomials are defined as $h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)$, $n = 0, 1, 2, ..$ and thus Hermite functions $\xi_n(x)$ are also defined as:

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(x),\ n = 1, 2, 3, ...$$

We can easily see that the Hermite functions $\xi_n(x)$ $n = 1, 2, 3, ...$ constitute an orthonormal basis in $L^2\left(\mathbb{R}^d\right)$ with respect to the weight $e^{-\frac{x^2}{2}}$.

Let now $\delta^j = \left(\delta_1^j, \delta_2^j, ..., \delta_d^j\right)$ where $\delta_i^j \in \mathbb{N}$ and assume the following tensor products

$$\xi_{\delta^j} := \xi_{\delta_1^j} \otimes \xi_{\delta_2^j} \otimes ... \otimes \xi_{\delta_d^j},\ j = 1, 2, 3, ...$$

where for $i < j$ inequality $\delta_1^i + \delta_2^i + ... + \delta_d^i \leq \delta_1^j + \delta_2^j + ... + \delta_d^j$ holds. The family of tensor products $\left\{\xi_{\delta^j}\right\}_{j=1}^{\infty}$ constitutes an orthogonal basis in $L^2\left(\mathbb{R}^d\right)$. We also introduce the countable multiindex via $I = \{a = (a_1, a_2, ...) | a_i \in \mathbb{N} \cup \{0\}\}$ for which there exists a finite number of $a_i \neq 0$. For each $a \in I$ we define stochastic Hermite polynomials H_a given by

$$H_a(\omega) = \prod_{i=1}^{\infty} h_{a_i}(\langle\omega, \xi_{\delta^i}\rangle),\ \omega \in \Omega.$$

We can see that H_a forms an orthogonal basis in $L^2(\Omega)$ and the norm $\|H_a\|$ satisfies

$$\|H_a\|^2_{L^2(\Omega)} = a! = a_1! a_2! ...$$

Theorem 1. *Every $f \in L^2(\Omega)$ has a unique Wiener- Ito chaos expansion in terms of stochastic Hermite polynomials, given by*

$$f(\omega) = \sum_{a \in I} c_a H_a(\omega),\ c_a \in \mathbb{R} \tag{2}$$

where

$$c_a = E(f(\omega)H_a(\omega)) = \int_{\Omega} f(\omega)H_a(\omega)dP(\omega).$$

In what follows we define the stochastic Hilbert space $(S)^{\rho,z,L^2(\Omega)}$, for $\rho \in [-1,1], z \in \mathbb{R}$, as the set of all sums

$$f = \sum_{a \in J} f_a H_a, f_a \in L^2(\Omega), \ \ \forall a \in I \tag{3}$$

with finite norm

$$\|f\|_{\rho,z,L^2(\Omega)} = \left(\sum_{a \in I} \|f_a\|^2_{L^2(\Omega)} (a!)^{1+\rho} (2\mathbb{N})^{za} \right)^{1/2}. \tag{4}$$

The norm given by (4) is induced by the inner product

$$(f,g)_{\rho,z,L^2(\Omega)} = \sum_{a \in I} (f_a, g_a)_{L^2(\Omega)} (a!)^{1+\rho} (2\mathbb{N})^{za}, \ f, g \in (S)^{\rho,z,L^2(\Omega)}$$

where

$$f = \sum_{a \in I} f_a H_a, \quad g = \sum_{a \in I} g_a H_a \tag{5}$$

and

$$(2\mathbb{N})^{za} := \prod_{j=1}^{\infty} (2j)^{za_j}.$$

Finally, we also define the usual Sobolev space $H_0^1(D)$ given by,

$$H_0^1(D) := \Big\{ v \in H^1(D) \, and \ v = 0 \ on \, \partial D \Big\}.$$

3. The Stochastic Helmholtz Boundary Value Problem

In this section we present the construction of an infinite hierarchy of deterministic equations for the stochastic Helmholtz equation. Furthermore, we study the well-posedness of our stochastic problem through the existence and uniqueness for each solution of the hierarchy of deterministic problems.

We consider the following stochastic boundary value problem

$$\Delta u + k^2 u = f \quad \text{in } D \tag{6}$$

$$u = g, \quad \text{on } \partial D \tag{7}$$

where f is a generalized stochastic source, g a stochastic boundary condition and $I = \{a = (a_1, a_2, ...) | a_i \in \mathbb{N} \cup \{0\}\}$ as given above (see page 2). For the stochastic problem (6)-(7) we use relations given in (5) as well as $u = \sum_a u_a H_a$, in order to get the infinite hierarchy of deterministic problems

$$\Delta u_a + k^2 u_a = f_a \quad \text{in } D \ \text{ and } \quad u_a = g_a \quad \text{on } \partial D \tag{8}$$

For the above deterministic problems we can get their corresponding variational formulations, and for the sake of brevity, we only give the variational formulation of the problem for $|a| = n$.

$$\Delta u_n + k^2 u_n = f_n \quad \text{in } D \tag{9}$$

$$u_n = g_n \quad \text{on } \partial D \tag{10}$$

given by

$$\alpha(u_n, v) = \ell(v) \ \forall v \in H^1(D). \tag{11}$$

In (11) the bilinear form $\alpha(u_n, v)$ on $H^1(D) \times H^1(D)$ is given by

$$\alpha(u_n, v) = \int_D \left(-\nabla u_n \cdot \nabla v + k^2 u_n v\right) dx \tag{12}$$

and the linear functional $\ell(v)$ on $H^1(D)$ by

$$\ell(v) = \int_D f_n v dx - \int_{\partial D} g_n v dx, \tag{13}$$

where the function $f_n \in L^2(D)$ and $g_n \in L^2(\partial D)$. In what follows we give the following proposition.

Proposition 1. *Let D be a bounded open subset of* $\mathbb{R}^d$ *,* $f_n \in L^2(D)$ *,*$g_n \in L^2(\partial D)$ *and* $k^2 \in L^\infty(D)$*, then the problem (11) has a unique solution* $u_n \in H^1(D)$*, which satisfies the following inequality*

$$\|u_n\|_{H^1(D)} \le c_n(\|f_n\|_{L^2(D)} + \|g_n\|_{H^{1/2}(\partial D)}) \tag{14}$$

The proof of the proposition uses the hypothesis of the Lax-Milgram theorem [7], and its omitted here for brevity. We now give the following main result.

Proposition 2. *If we define the weights* $w_a = (a!)^{1+\rho}(2\mathbb{N})^{za}$*,* $|a| = 0, 1, ..., n$*, then the stochastic problem (6)-(7) admits a unique weighted Wiener chaos solution* $u \in (S)^{\rho,z,L^2(D)}$.

Proof. From Proposition 1, each one of the deterministic problems (8) has a unique solution $u_a \in H^1(D)$ and via relation $u = \sum_a u_a H_a$ our stochastic problem (6)-(7) admits a unique solution. In relation (14), c_n depends on f_n, g_n and hence there is a positive constant c being the supremum of c_n, $n = 0, 1, 2, ...$ which satisfies inequality (14). Furthermore, if we raise each one of the inequalities (14) for $n = 0, 1, 2, ...$ to the square power, multiply both sides by the weights w_a, and add them we can get

$$\sum_{a\in I} w_a \|u_a\|^2_{H^1(D)} \le c^2 \sum_{a\in I} w_a \left(\|f_a\|^2_{H^1(D)} + \|g_a\|^2_{H^{1/2}(\partial D)}\right) \tag{15}$$

for a positive constant $c = Sup(c_n), n = 0, 1, 2, ...$ Using the fact that

$$\|u\|^2_{(S)^{\rho,z,L^2(D)}} = \sum_{a\in I} w_a \|u_a\|^2_{L^2(D)} \tag{16}$$

and taking into account that $\|u_\alpha\|_{L^2(D)} \le \|u_\alpha\|_{H^1(D)}$ via (15) we can easily get

$$\|u\|^2_{(S)^{\rho,z,L^2(D)}} \le c^2 \sum_{a\in I} w_a \left(\|f_a\|^2_{H^1(D)} + \|g_a\|^2_{H^{1/2}(\partial D)}\right) < \infty. \tag{17}$$

We also remark that an analogous estimation for the solution as in (17) is also valid in the space $(S)^{\rho,z,H^1(D)}$. □

4. Stochastic Boundary Data for Navier Equation

In this section we study the stochastic boundary value problem for Navier equation. Initially, similar to the acoustic case we construct an infinite hierarchy of deterministic problems and establish the well-posedness of our stochastic problem via the uniqueness and existence of each deterministic one. Let $D \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial D \equiv \Gamma$ being *Lipschitz*. Throughout this paper $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\mathbf{r})$ denotes the outward

unit normal vector at the point $\mathbf{r} \in \Gamma$. The problem is formulated as follows:
Find a vector function $\mathbf{u} \in (S)^{\rho,z,[L^2(D)]^2}$ *such that*

$$\Delta^* \mathbf{u}(\mathbf{r}) + \varrho\, \omega^2 \mathbf{u}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in D, \tag{18}$$

$$\mathbf{u}(\mathbf{r}) = \mathbf{g} := \sum_{\alpha} \mathbf{g}_\alpha H_\alpha, \quad \mathbf{r} \in \Gamma, \tag{19}$$

where the explicit expression for Δ^*, is given by

$$\Delta^* \mathbf{u}(\mathbf{r}) := \mu\, \Delta \mathbf{u}(\mathbf{r}) + (\,\lambda + \mu\,)\, \nabla\, \nabla \cdot\, \mathbf{u}(\mathbf{r}) \tag{20}$$

with $\omega \in \mathbb{R}$ in (18) denotes now the so called angular frequency, λ, μ are the Lamé constants and ϱ is the mass density. Since any element of the space $(S)^{\rho,z,[L^2(D)]^2}$ admits a Wiener chaos expansion [3,5], substituting the projections $\mathbf{u}_\alpha$ of $\mathbf{u}$ on H_α into the relation $\mathbf{u}(\mathbf{r}) = \sum_\alpha \mathbf{u}_\alpha H_\alpha$ we can construct the solution $\mathbf{u}$. We transform our stochastic problem into an infinite hierarchy of deterministic problems and we exploit uniqueness and existence results for each one [4]. Via the projections $\mathbf{u}_\alpha$, $\alpha \in I$ we get the following hierarchy of problems:

$$\Delta^* \mathbf{u}_\alpha(\mathbf{r}) + \varrho\, \omega^2 \mathbf{u}_\alpha(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in D, \tag{21}$$

$$\mathbf{u}_\alpha(\mathbf{r}) = \mathbf{g}_\alpha, \quad \mathbf{r} \in \Gamma, \tag{22}$$

For the above deterministic problems we can get their corresponding variational formulations, and for the sake of brevity, we only give the variational formulation of the problem for $|a| = n$ (21)-(22):
Find a solution $\mathbf{u}_n \in \left[H^1(\overline{D})\right]^2$ *such that*

$$\alpha(\mathbf{u}_n,\, \mathbf{v}) = \ell(\mathbf{v}), \textit{ for every } \mathbf{v} \in [H^1(D)]^2 \tag{23}$$

where the bilinear form $\alpha(\mathbf{u}_n,\, \mathbf{v})$ *on* $[H^1(D)]^2 \times [H^1(D)]^2$ *is given by*

$$\begin{aligned} \alpha(\mathbf{u}_n,\, \mathbf{v}) \quad &= \quad -\mu \int_D (\nabla \mathbf{u}_n) : (\nabla \bar{\mathbf{v}})\, dr - (\lambda + \mu) \int_D (\nabla \cdot \mathbf{u}_n)(\nabla \cdot \bar{\mathbf{v}})\, dr \\ &\quad + \int_D \rho \omega^2 \mathbf{u}_n \cdot \bar{\mathbf{v}}\, dr \end{aligned} \tag{24}$$

and the linear functional $\ell(\mathbf{v})$ *on* $[H^1(D)]^2$ *by*

$$\ell(\mathbf{v}) \quad = \quad -\mu \int_\Gamma \hat{\mathbf{n}} \cdot (\nabla \mathbf{g}_n) \cdot \bar{\mathbf{v}}\, ds - (\lambda + \mu) \int_\Gamma (\nabla \cdot \mathbf{g}_n) \hat{\mathbf{n}} \cdot \bar{\mathbf{v}}\, ds \tag{25}$$

Proposition 3. *Let* D *be an open subset of* $\mathbb{R}^2$ *and* $\mathbf{g}_n \in [L^2(D)]^2$, *then problem (21)-(22) is uniquely solvable and furthermore the solution* $\mathbf{u}_n \in [H^1(\overline{D})]^2$ *satisfies*

$$\|\mathbf{u}_n\|_{H^1(D)} \leq c\, \|\mathbf{g}_n\|_{H^{1/2}(\Gamma)} \textit{ for some positive constant } c. \tag{26}$$

In order now to establish now existence and uniqueness of (23) we need the following three lemmas for which their proofs are omitted here for brevity.

Lemma 1. *The bilinear form* $\alpha(\mathbf{u}_n,\, \mathbf{v})$ *is bounded i.e.*

$$|\alpha(\mathbf{u}_n,\, \mathbf{v})| \leq c_3 \|\mathbf{u}_n\|_{H^1(D)} \|\mathbf{v}\|_{H^1(D)}. \tag{27}$$

Lemma 2. *The following coercivity property for $\alpha(\mathbf{u}_n, \mathbf{u}_n)$ holds*

$$Re\{\alpha(\mathbf{u}_n, \mathbf{u}_n)\} \geq c \left\| \mathbf{u}_n \right\|^2_{H^1(D)}. \tag{28}$$

Lemma 3. *The linear functional $\ell(\mathbf{v})$ is bounded, i.e., there exists a positive constant c_1 such that*

$$|\ell(\mathbf{v})| \leq c_1 \left\| \mathbf{v} \right\|_{H^1(D)}. \tag{29}$$

The above procedure uses the hypothesis of the Lax-Milgram theorem [7] in order to derive the assertion of Proposition 3.

Proposition 4. *The stochastic problem (18)-(19) admits a unique Wiener chaos solution $\mathbf{u} \in (S)^{\rho,z,[L^2(D)]^2}$ that satisfies*

$$\left\| \mathbf{u} \right\|^2_{(S)^{\rho,z,[L^2(D)]^2}} \leq c^2 \sum_{\alpha} w_\alpha \left\| \mathbf{g}_\alpha \right\|^2_{H^{1/2}(\Gamma)} \quad \textit{where } w_\alpha = (a!)^{1+\rho} (2\mathbb{N})^{za},\ |\alpha| = 0, 1, 2, \ldots \tag{30}$$

Proof. We state here that each of the deterministic problems (21)-(22) admits a unique solution and we also mention that c_n depends on $\mathbf{g}_n$ and hence there is a positive constant c being the supremum of c_n, i.e. $c = sup\{c_n, n = 0, 1, 2, \ldots\}$. Thus the following inequalities hold:

$$\begin{aligned} \left\| \mathbf{u}_0 \right\|_{H^1(D)} &\leq c \left\| \mathbf{g}_0 \right\|_{H^{1/2}(\Gamma)} \\ \left\| \mathbf{u}_1 \right\|_{H^1(D)} &\leq c \left\| \mathbf{g}_1 \right\|_{H^{1/2}(\Gamma)} \\ &\vdots \\ \left\| \mathbf{u}_n \right\|_{H^1(D)} &\leq c \left\| \mathbf{g}_n \right\|_{H^{1/2}(\Gamma)} \\ &\vdots \end{aligned} \tag{31}$$

Raising these inequalities to the second power, multipling both sides of each inequality by w_α, adding them, and taking into account $\|u_\alpha\|_{[L^2(D)]^2} \leq \|u_\alpha\|_{[H^1(D)]^2}$ we get

$$\sum_{\alpha} w_\alpha \|\mathbf{u}_a\|^2_{L^2(D)} \leq c^2 \sum_{\alpha} w_\alpha \left\| \mathbf{g}_\alpha \right\|^2_{H^{1/2}(\Gamma)}. \tag{32}$$

Hence we easily arrive at

$$\left\| \mathbf{u} \right\|^2_{(S)^{\rho,z,[L^2(D)]^2}} \leq c^2 \sum_{\alpha} w_\alpha \left\| \mathbf{g}_\alpha \right\|^2_{H^{1/2}(\Gamma)} < \infty. \tag{33}$$

An analogous estimation for the solution, as in (33) is also valid in the space $(S)^{\rho,z,[H^1(D)]^2}$. □

5. Conclusions

In this paper well posedness of solutions for stochastic boundary value problems due to Helmholtz as well as Navier equation were established, via the study of their corresponding hierarchies of deterministic problems. Uniqueness, existence and regularity issues were addressed and we also make the following remarks:

(i) For the stochastic Helmholtz equation, with stochastic source and stochastic boundary condition, we proved that stochastic problem (6)-(7) admits a unique weighted Wiener chaos solution.

(ii) In the case of stochastic boundary data for Navier equation, a unique Wiener chaos solution for stochastic problem (18)-(19) was proved.

(iii) The proposed method can also be extended to cover the case of stochastic boundary value problem where the randomness is present in the equation (e.g. in k for the Helmholtz equation, or ϱ, λ, μ for the Navier equation) as well as in the boundary condition. The study of such cases in under progress and will be communicated separately.

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