computer sciences and mathematics forum

MDPI

Proceeding Paper

A First-Order Evolution Problem with Maximal Monotone Operators †

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† Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.

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Abstract: In Hilbert spaces, we are interested in proving the existence of bounded variation continuous solutions to a first-order evolution problem involving time and state dependent maximal monotone operators with perturbations. This new result is established, under a compactness assumption on the domain of the operators. As an example, we provide the corresponding existence result for the perturbed sweeping process.

Keywords: evolution problem; maximal monotone operator; bounded variation continuous; sweeping process

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Citation: Nadjiba, A.; Soumia, S. A First-Order Evolution Problem with Maximal Monotone Operators. *Comput. Sci. Math. Forum* **2023**, *1*, 0. https://doi.org/

Academic Editor: Firstname Lastname

Published: 28 April 2023

1. Introduction

The main objective of this paper is to study the following first-order evolution problem involving time and state dependent maximal monotone operators

$$\begin{cases} -\dfrac{du}{dr}(t) \in B(t,u(t))u(t) + f(t,u(t)) \quad \mathrm{dr}-\text{a.e. } t \in I := [0,T], \\ u(t) \in D(B(t,u(t))), \quad t \in I \end{cases} \tag{1}$$

where $B(t,x)$ is a maximal monotone operators defined on a Hilbert space H of bounded continuous variation (BV shortly) in state and Lipschitz in time, in the sense of pseudo-distance [1].

This problem have been studied in [2] with $dr = dt$ (Lebesgue measure). Differential inclusions with maximal monotone operator has been considered by many authors, see e.g., [3–5]. This paper contains three sections. In the first one we give the notation needed in our study. In the next section, we present the main existence result. In the last one we give an application.

2. Background Material

In this section, we introduce notations and definitions which we will be use it in all the paper:

Let H be a real separable Hilbert space whose inner product is denoted by $\langle\cdot,\cdot\rangle$ and the associated norm by $||\cdot||$ and $I := [0,T]$ $(T > 0)$ is an interval of $\mathbb{R}$.

Note by $\mathcal{C}_H(I)$ continuous maps x from I to H and consider the norm of uniform convergence on I $||x||_\infty = \sup_{t\in I} ||x(t)||$.

By $\mathcal{L}(I)$ (resp. $\mathcal{B}(H)$) the σ-algebra of measurable sets of I (resp. Borel σ-algebra of measurable sets of H) and $\overline{B}_H(x,r)$ the closed ball of center x and radius r on H, and by $\overline{B}_H$ the closed unit ball of H.

By $L^p_H(I)$ for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I ||x(t)||^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $||x||_{L^p_H(I)} = (\int_I ||x(t)||^p dt)^{\frac{1}{p}}$, $1 \leq p < +\infty$ (resp. endowed with the usual essential supremum norm $|| \cdot ||_{L^\infty_H(I)}$).

By $W^{1,2}(I, H)$, we denote the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$.

In second part we present some definitions and properties of maximal monotone operators. First we give definitions of domain, graph and rang see [6].

Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator

$$\mathrm{D}\,(A) = \{x \in H : \; Ax \neq \emptyset\},$$

$$R(A) = \{y \in H : \; \exists\, x \in \mathrm{D}\,(A),\; y \in Ax\} = \cup\{Ax : \; x \in \mathrm{D}\,(A)\},$$
$$Gr(A) = \{(x, y) \in H \times H : \; x \in \mathrm{D}\,(A),\; y \in Ax\}.$$

The operator $A : D(A) \subset H \rightrightarrows H$ is said to be monotone, if for $(x_i, y_i) \in Gr(A)$, $i = 1, 2$ one has $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$.

It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H denotes the identity map of H.

If A is a maximal monotone operator for every $x \in \mathrm{D}\,(A)$, Ax is nonempty, closed and convex. Then, we can define the projection of the origin into Ax, denote by A^0x.

the resolvent of A define by $J^A_\lambda = (I_H + \lambda A)^{-1}$ and the Yosida approximation of A by $A_\lambda = \frac{1}{\lambda}(I_H - J^A_\lambda)$, for $\lambda > 0$. These operators are both single-valued and defined on the whole space H. Recall that,in the sense of convex analysis, the normal cone to the set $C(t, x)$ at $v \in H$, for all $(t, x) \in I \times H$ is the subdifferential of the indicator function of the set at v that is

$$N_{C(t,x)} = \partial\delta_C(t, x) = \{\zeta \in H; \langle\zeta, z - y\rangle \leq 0 \;\forall z \in C(t, x)\}$$

which is maximal monotone operator with $\mathrm{D}\,(N_{C(t,x)}) = C(t, x)$

Proposition 1. *Let $A : D(A) \subset H \rightrightarrows H$ be a operator maximal monotone. Then, one has*

$$J^A_\lambda x \in \mathrm{D}\,(A) \;\; \textit{and} \;\; A_\lambda x \in A(J^A_\lambda x), \textit{ for every } \; x \in H. \tag{2}$$

Definition 1. *Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators. Then the pseudo-distance between A and B denoted by* $\mathrm{dis}\,(A, B)$ *(see [1]) is defined by*

$$\mathrm{dis}\,(A, B) = \sup\left\{\frac{\langle y_1 - y_2, x_2 - x_1\rangle}{1 + ||y_1|| + ||y_2||} : \; (x_1, y_1) \in Gr(A),\; (x_2, y_2) \in Gr(B)\right\}.$$

Clearly, $\mathrm{dis}\,(A, B) \in [0, +\infty]$, $\mathrm{dis}\,(A, B) = \mathrm{dis}\,(B, A)$ and $\mathrm{dis}\,(A, B) = 0$ iff $A = B$.

3. Main Results

In this section, we study the existence of bounded variation continuous solutions to the problem (1).

Theorem 1. *Assume that for any $(t, y) \in I \times H$, $B(t, y) : D(B(t, y)) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying*
(H^1_B). There exist a non-negative real constant $\lambda < \frac{2}{3}$, and a function $r : I \to [0, +\infty[$ which is continuous on $[0, T]$ and non-decreasing with $r(T) < \infty$ such that

$$\mathrm{dis}\,(B(t, y), B(s, z)) \leq r(t) - r(s) + \lambda\|y - z\|, \text{ for } 0 \leq s \leq t \leq T, \text{ for } y, z \in H.$$

(H_B^2) *There exists a non-negative real number c such that*

$$\|B^0(t,y)z\| \le c(1 + \|y\| + \|z\|) \text{ for } t \in I,\ y \in H,\ z \in D(B(t,y)).$$

(H_B^3) *For any bounded subset F of H, the set $D(B(I \times F))$ is relatively ball-compact.*
Let $f : I \times H \to H$ be a map such that

(1) *for any fixed $x \in H$, $f(\cdot, x)$ is measurable on I and for any fixed $t \in I$, $f(t, \cdot)$ is continuous on H;*
(2) *there exists a non-negative real constant L such that*

$$f(t,x) \le L(1 + ||x||) \text{ for all } (t,x) \in I \times H. \tag{3}$$

Then, for any $u_0 \in D(B(0,u_0))$, the evolution problem (1) admits a BV continuous solution $u : I \to H$.

Furthermore, $u'(t) = \frac{du}{dr}(t)$ with respect to dr, and

$$||u'(t)|| \le k \ \ dr - \text{a.e. } t \in I,$$

and

$$\|u(t) - u(s)\| \le k\Big(r(t) - r(s)\Big) \text{ for } 0 \le s \le t \le T,$$

for a real positive constant k depending on c, T, λ, $r(T)$.

Proof. Let us construct the sequences (u_n):
For any $n \ge 1$, define a partition of $I := [0, T]$ with

$$0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

For any $n \ge 1$ and $i = 0, 1, \cdots, n-1$, set

$$r_{i+1}^n = r(t_{i+1}^n) - r(t_i^n), \quad , r_i^n \le r_{i+1}^n \le k_n = \frac{r(T)}{n}, \tag{4}$$

Fix any $n \ge 1$. Put $u_0^n = u_0 \in D(B(0,u_0))$. For $i \in \{0, \cdots, n-1\}$ set

$$u_{i+1}^n = J_{i+1}^n\Big(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) dr(s)\Big),$$

then by (2) one gets $u_{i+1}^n \in D(B(t_{i+1}^n, u_i^n))$ and one writes

$$-\frac{1}{r_{i+1}^n}\Big(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(\zeta, u_i^n) dr(\zeta)\Big) \in B(t_{i+1}^n, u_i^n) u_{i+1}^n. \tag{5}$$

After some computation, we get

$$||u_{i+1}^n - u_i^n|| \le r_{i+1}^n \eta \sum_{j=0}^{i} \mu^j (1 + ||x_{i-j}^n|| + ||u_{i-j-1}^n||). \tag{6}$$

where μ and η are non negative real constants depending on L, c, λ. Thus, for any n and $i = 0, \cdots, n-1$ one obtains

$$||u_{i+1}^n|| \le \Big(||u_0|| + \eta \frac{r(T)}{1-\mu}\Big) \exp(\frac{3\eta r(T)}{1-\mu})\Big) = k_1. \tag{7}$$

This, along with (6) yields

$$||u^n_{i+1} - u^n_i|| \leq \frac{\eta(1+2k_1)}{1-\mu} r^n_{i+1} = k_2 r^n_{i+1}. \tag{8}$$

Define by $f_n(t) = f(t, u_n(\theta_n(t)))$ for any n and $t \in I$.

For any $n \geq 1$, define the sequence of maps u_n for all $t \in [t^n_i, t^n_{i+1}[, i \in \{0, \cdots, n-1\}$ by

$$u_n(t) = u^n_i + \frac{r(t) - r(t^n_i)}{r(t^n_{i+1}) - r(t^n_i)} \left(u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f_n(s) dr(s) \right) - \int_{t^n_i}^{t} f_n(s) dr(s),\ u_n(T) = u^n_n. \tag{9}$$

and then, set for any $n \geq 1$

$$\theta_n(t) = \begin{cases} t^n_i & \text{if } t \in]t^n_i, t^n_{i+1}] \quad \text{for some } i \in \{0, 1, \cdots, n-1\}, \\ 0 & \text{if } t = 0, \end{cases}$$

$$\phi_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t^n_{i+1} & \text{if } t \in]t^n_i, t^n_{i+1}] \quad \text{for some } i \in \{0, 1, \cdots, n-1\}. \end{cases}$$

by derivation one gets for all $t \in]t^n_i, t^n_{i+1}[$

$$\frac{du_n}{dr}(t) = \frac{1}{r^n_{i+1}} \left(u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f(\zeta, u^n_i) dr(\zeta) \right) - f(\zeta, u^n_i), \tag{10}$$

Hence, by (5) for each $n \in \mathbb{N}^*$, there is a null Lebesgue measure set $K_n \subset I$ such that

$$-\frac{du_n}{dr}(t) \in B(\theta_n(t), u_n(\theta_n(t))) u_n(\phi_n(t)) + f_n(t) \text{ dr} - \text{a.e } t \in I, \tag{11}$$

$$u_n(\theta_n(t)) \in D(B(\theta_n(t), u_n(\theta_n(t)))),\ t \in I. \tag{12}$$

In view of (4), (8) and (10) one has for all $t \in [t^n_i, t^n_{i+1}[$

$$\|\frac{du_n}{dr}(t)\| \leq 2k_2 + 2L(1+k_1) = k$$

Thanks to (12) and (7), one gets for all $t \in I$

$$(u_n(\theta_n(t))) \subset D(B(I \times k_1 \overline{B}_H)).$$

The latter inclusions along with (H^3_B) entail that the set $\{u_n(\theta_n(t)) : n \in \mathbb{N}^*\}$ is relatively compact in H and by the absolute continuity of u_n for any $n \in \mathbb{N}$, the set $\{u_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in H.

Note that $\{u_n(\cdot) : n \in \mathbb{N}^*\}$ is equicontinuous. By Ascoli's theorem, $(u_n(\cdot))_n$ is relatively compact in $\mathcal{C}_H(I)$. then

$$\|u_n(\theta_n(t)) - u(t)\| \to 0, as \quad n \to \infty. \tag{13}$$

Remark that u_n is continue and $(u_n(t))$ converge to $u(t)$ then

$$\langle e, u(t) - u(s) \rangle = \left\langle e, \int_s^t z(\tau) d\tau \right\rangle.$$

Hence, given any $s, t \in I$ with $s \leq t$, we get $\int_s^t z(\tau)d\tau = u(t) - u(s)$, then $u(\cdot)$ is absolutely continuous. moreover $z(\cdot)$ coincides almost everywhere in I with $\frac{du}{dr}(\cdot)$, and it results

$$\frac{du_n}{dr} \to \frac{du}{dr} \text{ weakly in } L^2(I, H, dr). \tag{14}$$

we easily prove that

$$f_n(\cdot) \to f(\cdot, u(\cdot)) \tag{15}$$

Now, let us establish the inclusion

$$u(t) \in D(B(t, u(t))), \quad t \in I,$$

$$u(0) = u_0 \in D(B(0, u_0)).$$

we have that $u_n(\theta_n(t)) \in D(B(\theta_n(t), u_n(\theta_n(t))))$ for all $t \in I$ (see (12)). Combining (H_B^1) with (4), (4) yields

$$\text{dis}\,(B(\theta_n(t), u_n(\theta_n(t))), B(t, u(t))) \to 0 \text{ as } n \to \infty. \tag{16}$$

In view of (H_B^2), (13), one deduces that $(w_n) = (B^0(\theta_n(t), u_n(\theta_n(t)))u_n(\theta_n(t)))$ is bounded. Then, we may extract from (w_n) a subsequence that weakly converges to $w \in H$. Since the sequence $(u_n(\theta_n(t)))$ converges to $u(t)$ in H then $u(t) \in D(B(t, u(t)))$, $t \in I$.

Next, let us show the differential inclusion

$$-\frac{du}{dr}(t) \in B(t, u(t))u(t) + f(t, u(t)) \quad \text{dr} - \text{a.e. } t \in I.$$

It suffices to show that

$$\langle \frac{du}{dr}(t) + f(s, u(t)), u(t) - z\rangle \leq \langle B^0(t, u(t))z, z - u(t)\rangle \quad \text{dr} - \text{a.e. } t \in I,$$

by (14), (15) one deduces that $(\frac{du_n}{dr} + f_n(\cdot))$ weakly converges to $\frac{du}{dr}(\cdot) + f(\cdot, u(\cdot))$ in $L^2(I, H, dr)$. Then, there exists a sequence (ξ_j) such that for each $j \in \mathbb{N}$, $\xi_j \in co\{\frac{du_k}{dr} + f_k(\cdot),\ k \geq j\}$ and (ξ_j) strongly converges to $\frac{du}{dr}(\cdot) + f(\cdot, u(\cdot))$ in $L^2(I, H, dr)$. In other words, there exists a subset K of I with null-Lebesgue measure and a subsequence (j_p) of $\mathbb{N}$ such that for all $t \in I \setminus K$, $(\xi_{j_p}(t))$ converges to $\frac{du}{dr}(t) + f(t, u(t))$. Hence, for $t \in I \setminus K$

$$\frac{du}{dr}(t) + f(t, u(t)) \in \bigcap_{p \in \mathbb{N}} \overline{co}\{\frac{du_k}{dr}(t) + f_k(t),\ k \geq j_p\},$$

then $t \in I \setminus K$ and any $w \in H$

$$\langle \frac{du}{dr}(t) + f(t, u(t)), w\rangle \leq \limsup_{n \to \infty} \langle \frac{du_n}{dr}(t) + f_n(t), w\rangle.$$

by (16) ensures the existence of a sequence (z_n) such that $z_n \in D(B(\theta_n(t), u_n(\theta_n(t))))$

$$z_n \to z \text{ and } B^0(\theta_n(t), u_n(\theta_n(t)))z_n \to B^0(t, u(t))z.$$

For $n \geq 1$, let $I \setminus K_n$ denote the set on which (11) holds. Since $B(t, y)$ is monotone for any $(t, y) \in I \times H$, one obtains for $t \in I \setminus K_n$

$$\langle \frac{du_n}{dr}(t) + f_n(t), u_n(\theta_n(t)) - z_n\rangle \leq \langle B^0(\theta_n(t), u_n(\theta_n(t)))z_n, z_n - u_n(\theta_n(t))\rangle.$$

then

$$\langle \quad \frac{du_n}{dr}(t) + f_n(t), u(t) - z\rangle \leq \langle B^0(\theta_n(t), u_n(\theta_n(t)))z_n, z_n - u_n(\theta_n(t))\rangle$$
$$+\Big(k_1 + L(1+k_1)\Big)\Big(\|u_n(\theta_n(t)) - u(t)\| + \|z_n - z\|\Big).$$

which give

$$\limsup_{n\to\infty}\langle\frac{du_n}{dr}(t) + f_n(t), u(t) - z\rangle \leq \langle B^0(t, u(t))z, z - u(t)\rangle.$$

The differential inclusion

$$-\frac{du}{dr}(t) \in B(t, u(t))u(t) + f(t, u(t)) \quad \mathrm{dr} - \text{a.e. } t \in I,$$

therefore holds true.

4. Sweeping Process

As an application of Theorem (1) we give the following result

Theorem 2. *Let $C : I \times H \rightrightarrows H$ be a set-valued mapping satisfying:*
(H_1') For each $t \in I$, $C(t, y)$ is a non-empty closed convex subset of H.
(H_2') There exist a non-negative real constant $\lambda < \frac{2}{3}$, and a function $r : I \to [0, +\infty[$ which is continuous on $[0, T[$ and non-decreasing with $r(T) < \infty$ and $r(0) = 0$ such that

$$|d(x, C(t, u)) - d(x, C(s, v))| \leq |r(t) - r(s)| + \lambda||v - u|| \; \forall t, s \in I, \;\; \forall x, v, u \in H.$$

(H_3') For any bounded subset $F \subset H$, the set $C(I \times F)$ is relatively ball compact.
Let $f : I \times H \to H$ be a map such that satisfying (1), (2), then for any $u_0 \in C(0, u_0)$ the sweeping process

$$\begin{cases} -\dfrac{du}{dr}(t) \in N_{C(t,u(t))}u(t) + f(t, u(t)) \quad \mathrm{dr}\,\text{a.e. } t \in I, \\ u(t) \in C(t, u(t)), \quad t \in I \\ u(0) = u_0 \in C(0, u_0); \end{cases}$$

has a BV solution $u : I \to H$.

Author Contributions:

Funding:

Institutional Review Board Statement:

Informed Consent Statement:

Data Availability Statement:

Conflicts of Interest:

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