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Proceeding Paper

The Desymmetrized PSL(2, Z) Group; Its 'Square-Box' One-Cusp Congruence Subgroups †

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Abstract: The desymmetrized PSL(2, **Z**) group is studied. The Fourier coefficients of the non-holomorphic one-cusp Eisenstein series expansion are summed; as a new further result, a new dependence on the Euler's γ constant is found. The congruence subgroups of the desymmetrized $PSL(2,\mathbf{Z})$ are scrutinized, and the related structures are investigated. A 'square-box' one-cusp congruence subgroup is constructed. New leaky tori are constructed.

Keywords: desymmetrized modular group; non-holomorphic Eisenstein series summation; number theory; congruence subgroups

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1. Introduction

The desymmetrized $PSL(2,\mathbf{Z})$ group is considered.

In [1], the motivations of the choice of the desymmetrized group are given, as it contains the parabolic element.

In [2] p. 76 and pp. 508–540, The Fourier expansion of the non-holomorphic Eisenstein cusp series of the desymmetrized $PSL(2,\mathbf{Z})$ group is developped.

The well-posed-ness of the summation of the Fourier coefficients is assured after the control of the meromorphic continuability of the $\zeta_k(s)$ functions, as in [3]; the further complements of the meromorphic continuability, as well as the controls on the summability of the series, from the points of view both analytical and of the number-theoretical issues, are briefly recapitulated in [4].

The construction of leaky tori is proposed in [5–7]. Further results are obtained in [8].

In the present paper, the summation of the Fourier coefficients on the non-holomorphic Eisenstein series of the one-cusp desymmetrized $PSL(2,\mathbf{Z})$ group is achieved [4], and a new dependence on the Euler γ constant is found.

The congruence subgroups of the desymmetrized $PSL(2,\mathbf{Z})$ group are newly constructed; new constructions of the leaky torus are presented after these constructions [9]. It is important to recall that the (sub)-grouppal structures have a role in the study of the modular Monster group [10].

2. Materials and Methods

In [11], the use of the divisor function in the faltungs of the Fourier coefficients in the Eisenstein-Maass series is outlined.

In [2] Ch. 11.4, the distribution of the Fourier coefficients of the non-holomorphic one-cusp form of the desymmetrized $PSL(2,\mathbf{Z})$ is found as the Fourier expansion of the Eisenstein series $E(z;s)$

$$E(z,s) = y^s + \phi(s)y^{1-s} + \sum_{1\leq m<+\infty} \sqrt{y}\phi_m(s)K_{s-\frac{1}{2}}(2\pi \mid m \mid y)e^{2\pi imx} \tag{1}$$

where the coefficients $\phi(s)$ and $\phi_n(s)$ are defined in [2] in p. 508 as

$$\phi(s) \equiv \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\frac{\zeta(2-2s)}{\zeta(2s)}, \quad (2)$$

and in p. 76 the following specification is spelled out

$$\phi_m(s) \equiv 2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\sum_{j=1}^{j=\infty}\frac{c_j(m)}{k^2}, \quad (3)$$

where the latter term is summed for the considered grouppal structure as

$$\phi_m(s) = 2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\frac{\sigma_{1-2s}[\mid m \mid]}{\zeta(2s)}. \quad (4)$$

The non-holomorphic Eisenstein series $E(z,s)$ therefore becomes

$$E(z,s) = \sum_{n=-\infty}^{n+\infty} a_n(y,s)e^{2\pi inx}, \quad (5)$$

where

$$a_0(y,s) = \pi^{-s}\gamma(s)\zeta(2s) + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}, \quad (6)$$

and

$$a_n(y,s) = 2\mid n\mid^{s-\frac{1}{2}}\sigma_{1-2s}(\mid n\mid)\sqrt{y}K_{s-\frac{1}{2}}(2\pi\mid n\mid y). \quad (7)$$

3. Results

The Eisenstein series Equation (1) is summed after summing the Fourier coefficients Equation (4) after applying the Dirichlet formula, after which one now here thus for the terms following Equation (5)

$$\sigma_{1-2s}[\mid m\mid] = \mid m\mid^{\frac{1}{2}-s}\sigma_s[\mid m\mid] \quad (8)$$

and, for $s \geq 1$,

$$\sigma_{1-2s}[\mid m\mid] = \mid m\mid^{\frac{1}{2}-s}\left[s\log(s) + (2\gamma-1)s + O(\sqrt{s})\right] \quad (9)$$

γ being the Euler's γ constant.

Formula Equation (4) is now summed as

$$\phi_\gamma(s) = 2\pi^s\frac{1}{\Gamma(s)}\frac{\left[s\log(s) + (2\gamma-1)s + O(\sqrt{s})\right]}{\zeta(2s)}: \quad (10)$$

the dependence of the term $\phi_m(s)$ in Equation (4) is now summed as with a new dependence on the Euler's γ constant.

4. Applications: The Congruence Subgroups and the New Leaky Tori

4.1. The Congruence Subgroup Γ_0 of $PSL(2, \mathbf{Z})$

The 'square-box' congruence subgroup Γ_0 of $PSL(2, \mathbf{Z})$ is constructed on the one-cusp 'square-box' domain

$$a : x = 0, \tag{11a}$$
$$b_1 : x = -1, \tag{11b}$$
$$c : x^2 + y^2 = 1, \tag{11c}$$
$$d_1 : (x+1)^2 + y^2 = 1, \tag{11d}$$

and generated after the (hyperbolic) reflections

$$R_1 : z \to z' \equiv -\frac{1}{\bar{z}}, \tag{12a}$$
$$T_{-1} : z \to z' \equiv -\bar{z} + 2, \tag{12b}$$
$$T_0 : z \to z' \equiv -\bar{z}, \tag{12c}$$
$$R_2 : z \to z' = T_{\frac{1}{2}} R_1 T_{-\frac{1}{2}} z \tag{12d}$$

After the Markoff uniqueness property [12], there exists an isometry between any two simple closed geodesics of equal length on a torus; furthermore, the Laplace-Beltrami operator on Riemann surfaces of constant negative curvature is proven to have rigidity property. It is therefore possible to present new constructions of leaky tori.

4.1.1. A New Gutzwiller Leaky Tori

A leaky torus is defined in [5]; it is obtained after unfolding the $PSL(2, \mathbf{Z})$ [6] according to the triangular domains of the $PSL(2, \mathbf{Z})$ domain in a congruence subgroup of $PSL(2, \mathbf{Z})$ domain.

The leaky torus in [8] is equivalent to that of [6] p. 181 with respect to both the domain and the generators.

A new Gutzwiller leaky torus is here generated after the reflections

$$R_n : z \to z' \equiv T_{-n} R_1 T_n z, \quad x > 0, \tag{13a}$$
$$R_{-n} : z \to z' = T_n R_1 \tag{13b}$$

with

$$T_n : T_n z = -\bar{z} - 2n, \quad x > 0, \tag{14a}$$
$$T_n : T_n z = -\bar{z} + 2n, \quad x < 0, \tag{14b}$$

and

$$R_1 : R_1 z = -\frac{1}{\bar{z}} \tag{15}$$

on the domain of sides C_n, defined as

$$y \geq \sqrt{1 - (x-n)^2}, \quad n - \frac{1}{2} < x \leq n + \frac{1}{2}, \tag{16}$$

in the limit $n \to \infty$.

4.1.2. A New Leaky Torus from the Desymmetrized Triangle Group

It is possible to construct a leaky torus from the desymmetrized domain of the $PSL(2, \mathbf{Z})$ group. The leaky torus is thus constructed after the unfolding of the chosen trajectory according to the domain of the desymmetrized (triangular) $PSL(2,\mathbf{Z})$ group; the leaky torus is generated after the generators

$$T_{1,n} : z \to z' = T_{-\frac{1}{2}+n} R_1 T_{n-\frac{1}{2}}, \;\; n - \frac{1}{2} < x \leq n, \tag{17a}$$

$$T_{2,n} z \to z' = T_{-\frac{1}{2}+n} R_2 T_{n-\frac{1}{2}}, \;\; n < x \leq n + \frac{1}{2}, \tag{17b}$$

on the domain delimited after the sides C_n defined as

$$y = \sqrt{1 - \left(x - (n - \frac{1}{2})\right)^2}, \;\; n - \frac{1}{2} < x \leq n, \tag{18a}$$

$$y = \sqrt{1 - \left(x - (n + \frac{1}{2})\right)^2}, \;\; n < x \leq n + \frac{1}{2}. \tag{18b}$$

4.1.3. A New Leaky Torus from the 'Square-Box' Congruence Subgroup

A new leaky torus from the congruence subgroups of Γ_0 of $PSL(2,\mathbf{Z})$ is obtained here after unfolding [13,14] the 'square box' of the Γ_0 congruence subgroup of the desymmetrized $PSL(2,\mathbf{Z})$ group domain into the congruence subgroup $\Gamma_0(N)$ in the limit $N \to \infty$. This new leaky torus is defined on the domain of sides c_n, constructed as

$$y \geq \sqrt{1 - (x-n)^2}, \;\; n < x \leq n + \frac{1}{2}, \tag{19a}$$

$$y \geq \sqrt{1 - (x-(n+1))^2}, \;\; n + \frac{1}{2} < x \leq n+1, \tag{19b}$$

and generated after the reflections

$$T_{1,n+1} : z \to z' = T^{-1}_{n+1} R_1 T_{n+1} z, \;\; n + \frac{1}{2} < x \leq n+1, \tag{20a}$$

$$T_{1,n+1} : z \to z' = T^{-1}_{n+1} R_2 T_{n+1} z, \;\; n < x \leq n + \frac{1}{2} \tag{20b}$$

(which contain, of course, also the case R_1, R_2).

The new generators Equation (20) identify arcs of circumferences that are the sides of two different 'square boxes' congruence subgroups delimiting the domain of the congruence subgroups of the desymmetrized $PSL(2,\mathbf{Z})$ group.

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