The desymmetrized PSL(2, Z) group; its 'square-box' one-cusp congruence subgroups

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Abstract

The characterizing features of the **desymmetrized** $PSL(2,\mathbb{Z})$ **group** are investigated.

The **Fourier coefficients of the non-holomorphic one-cusp Eisenstein series** expansion are summed; a **new dependence on the Euler's** γ **constant** is spelled out.

The **new 'square-box' one-cusp congruence subgroup** of the desymmetrized $PSL(2,\mathbb{Z})$ are constructed. Related structures are studied.

New leaky tori are built.

Summary

The desymmetrized $PSL(2,\mathbb{Z})$ group

- summation of the non-holomorphic one-cusp Eisenstein series of the desymmetrized $PSL(2,\mathbb{Z})$ group:

• a new dependence on the Euler constant is found; - new 'square-box' congruence subgroups;

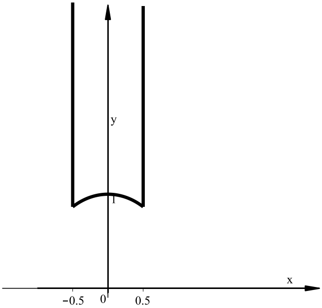
- new constructions of leaky tori.

Introduction

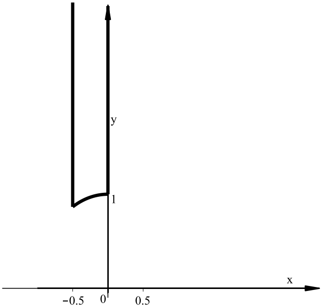
The properties of the *desymmetrized* $PSL(2,\mathbb{Z})$ *group* are studied. The *motivations* of this choice are given, as it contains the parabolic element.

The **Fourier coefficients of the non-holomorphic Eisenstein cusp series** are here **summed** after the control of the *meromorphic continuability* is recapitulated.

The **one-cusp 'square box' congruence subgroup of the desymmetrized** $PSL(2,\mathbb{Z})$ **is newly defined**.
Further congruence subgroups are newly found accordingly, after the which the **new construction** of **leaky tori** is obtained.



The domain of the group $PSL(2, \mathbb{Z})$ on the Upper Poincaré Half Plane (black solid line).



The domain of the desymmetrized group $PSL(2, \mathbb{Z})$ on the Upper Poincaré Half Plane (black solid line).

Control of the meromorphic continuability of the non-holomorphic one-cusp form of the desymmetrized $PSL(2,\mathbb{Z})$ group

The analysis is motivated after

Bykovskii, V. A. On a summation formula in the spectral theory of automorphic functions and its applications in analytic number theory. *Dokl. Akad. Nauk SSSR)*, **1982**, *264*, 275–277.

as the desymmetrized $PSL(2,\mathbb{Z})$ group contains the parabolic element; furthermore, the remainders for particular divisor functions related to the $\zeta_k(s)$ functions are calculated.

The use of the divisor function in the faltungs of the Fourier coefficients in the Eisenstein-Maass series is established.

Kuznetsov, N. V. Convolution of Fourier coefficients of Eisenstein-Maass series. *J. Sov. Math.*, **1985**, *29*, 1131–1159.

The absence of eigenvalues in the discrete spectrum within the interval $(0,-1/4]$ is controlled.

N. V. Kuznetsov, The Petersson hypothesis for parabolic forms of weight zero and the Linnik hypothesis. I. A sum of Kloosterman sums, Math. Sb., , No. 3, 334–383 (1980).

Distribution of the Fourier coefficients of the non-holomorphic one-cusp form of the desymmetrized $PSL(2,\mathbb{Z})$

Found as the Fourier expansion of the Eisenstein series $E(z;s)$

$$E(z,s)=y^s+\phi(s)y^{1-s}+\sum_{1\leq m<+\infty}\sqrt{y}\phi_m(s)K_{s-\frac{1}{2}}(2\pi\mid m\mid y)e^{2\pi imx} \quad (1)$$

- coefficients $\phi(s)$ and are defined as

$$\phi(s)\equiv\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}=\pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\frac{\zeta(2-2s)}{\zeta(2s)}, \quad (2)$$

with $\phi_m(s)$ specified as

$$\phi_m(s)\equiv 2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\sum_{j=1}^{j=\infty}\frac{c_j(m)}{k^2}, \quad (3)$$

where the latter term is summed for the considered grouppal structure as

$$\phi_m(s)=2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\frac{\sigma_{1-2s}[\mid m\mid]}{\zeta(2s)}. \quad (4)$$

The non-holomorphic Eisenstein series $E(z, s)$ therefore becomes

$$E(z,s) = \sum_{n=-\infty}^{n+\infty} a_n(y,s) e^{2\pi inx}, \qquad (5)$$

where

$$a_0(y,s) = \pi^{-s}\gamma(s)\zeta(2s) + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}, \qquad (6)$$

and

$$a_n(y,s) = 2 \mid n \mid^{s-\frac{1}{2}} \sigma_{1-2s}(\mid n \mid)\sqrt{y}K_{s-\frac{1}{2}}(2\pi \mid n \mid y). \qquad (7)$$

D. A. Hejhal, The Selberg Trace Formula for PSL (2,R): Volume 2 (Lecture Notes in Mathematics), Springer nature, Hemsbach (Germany) (1983).

Summation of the non-holomorphic Eisenstein series

Eisenstein series summed after **summing the Fourier coefficients** with the *Dirichlet formula*

$$\sigma_{1-2s}[\mid m \mid] = \mid m \mid^{\frac{1}{2}-s} \sigma_s[\mid m \mid] \tag{8}$$

and, for $s \geq 1$,

$$\sigma_{1-2s}[\mid m \mid] = \mid m \mid^{\frac{1}{2}-s} \left[s \log(s) + (2\gamma - 1)s + O(\sqrt{s}) \right] \tag{9}$$

γ being the Euler's γ constant.
The **non-holomorphic Eisenstein series is now summed as**

$$\phi_\gamma(s) = 2\pi^s \frac{1}{\Gamma(s)} \frac{\left[s \log(s) + (2\gamma - 1)s + O(\sqrt{s}) \right]}{\zeta(2s)} : \tag{10}$$

the dependence of the term $\phi_m(s)$ is now summed as with a **new dependence on the Euler's γ constant**.

O. M. Lecian, Summation of the Fourier coefficients of the non-holomorphic Eisenstein cusp series of the PSL(2, Z) group. *Int. Journ. of Math. and Computer Research*, **2023**, *11*, 3190–3194.

The congruence subgroups

The **'square-box' congruence subgroup** Γ_0 of $PSL(2,\mathbb{Z})$ is constructed on the **one-cusp 'square-box' new domain**

$$a : x = 0, \tag{11a}$$

$$b_1 : x = -1, \tag{11b}$$

$$c : x^2 + y^2 = 1, \tag{11c}$$

$$d_1 : (x+1)^2 + y^2 = 1, \tag{11d}$$

and generated after the **new (hyperbolic) reflections**

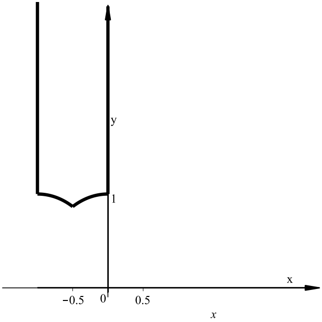
$$R_1 : z \rightarrow z' \equiv -\frac{1}{\bar{z}}, \tag{12a}$$

$$T_{-1} : z \rightarrow z' \equiv -\bar{z} + 2, \tag{12b}$$

$$T_0 : z \rightarrow z' \equiv -\bar{z}, \tag{12c}$$

$$R_2 : z \rightarrow z' = T_{\frac{1}{2}} R_1 T_{-\frac{1}{2}} z \tag{12d}$$

(12e)



The domain of the 'square box' congruence subgroup Γ_0 of the desymmetrized $PSL(2,\mathbb{Z})$ group on the Upper Poincaré Half Plane (black solid line).

New leaky tori

New Gutzwiller leaky tori

After the Markoff uniqueness property, there exists an isometry between any two simple closed geodesics of equal length on a torus;

McShane, G.; Parlier, H. Simple closed geodesics of equal length on a torus. In *Geometry of Riemann surfaces*; Gardiner, F. P., González-Diez, G., Kourouniotis, C., Eds.; Cambridge University Pres: Cambridge, UK; pp. 268–282.

furthermore, the Laplace-Beltrami operator on Riemann surfaces of constant negative curvature is proven to have rigidity property

Croke, C. B., Shrafutdinov, V. A., Spectral rigidity of a compact negatively curved manifold. *Topology*, **1998**, *37*, 1265–1273. .

It is therefore possible to present **new constructions of leaky tori**.

A leaky torus is defined in

Gutzwiller, M. C. Stochastic behavior in quantum scattering. *Physica D: Nonlin. Phen.*, **1983**, *7*, 341–355;

it is obtained after unfolding the $PSL(2,\mathbb{Z})$

Terras, A. *Harmonic analysis on symmetric spaces and applications, Vol. 1*; Springer-Verlag: New York, USA, 1985.

according to the triangular domains of the $PSL(2,\mathbb{Z})$ domain in a congruence subgroup of $PSL(2,\mathbb{Z})$ domain.

The leaky torus in

Chan, C. T.; Zainuddin, H.; Molladavoudi, S. Computation of Quantum Bound States on a Singly Punctured Two-Torus. *Chin. Phys. Lett.*, **2013**, *30*, 010304-1–010304-1-4.

is equivalent to the latter with respect to both the domain and the generators.

A new Gutzwiller leaky torus

A new Gutzwiller leaky torus is here generated after the **new reflections**

$$R_n : z \rightarrow z' \equiv T_{-n} R_1 T_n z, \quad x > 0, \tag{13a}$$

$$R_{-n} : z \rightarrow z' = T_n R_1 \tag{13b}$$

with

$$T_n : T_n z = -\bar{z} - 2n, \quad x > 0, \tag{14a}$$

$$T_n : T_n z = -\bar{z} + 2n, \quad x < 0, \tag{14b}$$

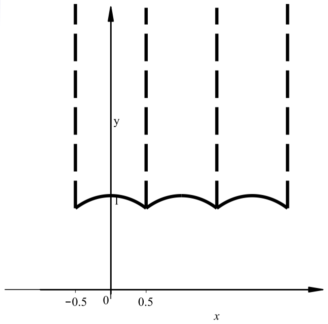
and

$$R_1 : R_1 z = -\frac{1}{\bar{z}} \tag{15}$$

on the **new domain** of sides $\mathcal{C}_n$, defined as

$$y \geq \sqrt{1-(x-n)^2}, \quad n - \frac{1}{2} < x \leq n + \frac{1}{2}, \tag{16}$$

in the limit $n \rightarrow \infty$.



The domain of a new Gutzwiller leaky torus after as a congruence subgroup of the symmetric $PSL(2, \mathbb{Z})$ group from the representation of the Terras congruence subgroup on the Upper Poincaré Half Plane (black solid line); the segments of degenerate geodesics ('vertical', dashed lines) are the symmetry lines with respect to which the generators of the hyperbolic reflections T act.

A new leaky torus from the desymmetrized triangle group

It is possible to construct a **new leaky torus from the desymmetrized domain of the $PSL(2, \mathbb{Z})$ group**.

The leaky torus is thus constructed after the **unfolding of the chosen trajectory according to the domain of the desymmetrized (triangular) $PSL(2, \mathbb{Z})$ group**.

The leaky torus is generated after the **new generators**

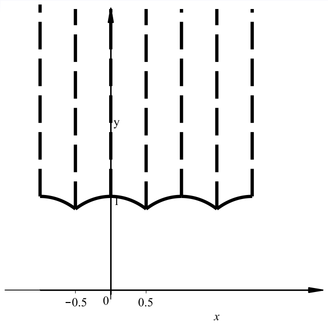
$$T_{1,n} : z \to z' = T_{-\frac{1}{2}+n} R_1 T_{n-\frac{1}{2}}, \quad n - \frac{1}{2} < x \leq n, \tag{17a}$$

$$T_{2,n} z \to z' = T_{-\frac{1}{2}+n} R_2 T_{n-\frac{1}{2}}, \quad n < x \leq n + \frac{1}{2}, \tag{17b}$$

on the domain delimited after the **new sides** C_n defined as

$$y = \sqrt{1 - \left(x - (n - \frac{1}{2})\right)^2}, \quad n - \frac{1}{2} < x \leq n, \tag{18a}$$

$$y = \sqrt{1 - \left(x - (n + \frac{1}{2})\right)^2}, \quad n < x \leq n + \frac{1}{2}. \tag{18b}$$



The domain of the leaky torus as a congruence subgroup of the desymmetrized $PSL(2,\mathbb{Z})$ group from the representation on the Upper Poincaré Half Plane (black solid line); the segments of degenerate geodesics ('vertical', dashed lines) are the symmetry lines with respect to which the generators of the hyperbolic reflections T act.

A new leaky torus from the 'square-box' congruence subgroup

A new leaky torus from the congruence subgroups of Γ_0 of $PSL(2,\mathbb{Z})$ is obtained here after unfolding the 'square box' of the Γ_0 congruence subgroup of the desymmetrized $PSL(2,\mathbb{Z})$ group domain into the congruence subgroup $\Gamma_0(N)$ in the limit $N \to \infty$.

This new leaky torus is defined on the **new domain** of sides c_n, constructed as

$$y \geq \sqrt{1-(x-n)^2}, \quad n < x \leq n+\frac{1}{2}, \tag{19a}$$

$$y \geq \sqrt{1-(x-(n+1))^2}, \quad n+\frac{1}{2} < x \leq n+1, \tag{19b}$$

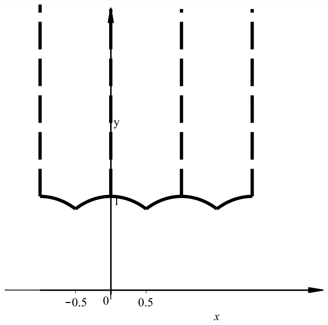
and generated after the new reflections

$$T_{1,n+1} : z \to z' = T_{n+1}^{-1} R_1 T_{n+1} z, \quad n+\frac{1}{2} < x \leq n+1, \tag{20a}$$

$$T_{1,n+1} : z \to z' = T_{n+1}^{-1} R_2 T_{n+1} z, \quad n < x \leq n+\frac{1}{2} \tag{20b}$$

(which contain, of course, also the case R_1, R_2),

The new generators identify arcs of circumferences that are the sides of two different 'square boxes' congruence subgroups delimiting the domain of the congruence subgroups of the desymmetrized $PSL(2,\mathbb{Z})$ group.



The domain of the leaky torus after as a congruence subgroup of the Γ_0 congruence subgroup of the desymmetrized $PSL(2, \mathbb{Z})$ group on the Upper Poincaré Half Plane (black solid line); the segments of degenerate geodesics ('vertical', dashed lines) are the symmetry lines with respect to which the generators of the hyperbolic reflections T act.

Outlook and perspectives

The desymmetrized $PSL(2, \mathbb{Z})$ group is considered.
The Fourier coefficients of the pertinent non-holomorphic one-cusp Eisenstein series are summed: a new dependence on the Euler γ constant is found.
The congruence subgroups of the desymmetrized $PSL(2, Z)$ group are studied.
New constructions of leaky tori are proposed.
The (sub)-grouppal structures have a role in the study of the modular Monster group.

Thank You for Your attention.