Article

Problem-Driven Teaching: Estimating the population from a sample

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Abstract: Today, mathematics teachers are realizing that there are many benefits of problem-driven teaching, but also face a number of challenges, such as a lack of confidence, not knowing how to design high-quality problems, and how to deal with the challenges that emerge in problem-driven lessons and statistics classes. To answer these questions, we propose a model for problem-driven teaching which concludes with three stages: The stage of preparation of the problems; The stage of implementation of teaching; The stage of evaluation and looking-back. Accordingly, we select the topic of 'Estimating the population from a sample' in the General Senior Secondary Textbook, Mathematics, Compulsory Book 2 to design a problem-driven instruction and as an application of problem posing. In the instruction, we will pose several interesting problems related to the topic. These problems can successfully stimulate students' interest in learning statistics. Through these problems, the maths teacher can guide the students to think actively and pose and solve the problem themselves, thus understanding the related concepts and methods in statistics. In addition, the design model applied here is a ready-to-use tool for teachers in designing specifically problem-driven instruction and also can improve teachers' problem-posing skills.

Keywords: problem-driven teaching; problem-posing; problem-solving; teaching statistics; instructional design

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1. Introduction

Problems are at the heart of mathematics and statistics. The 23 most influential mathematical problems posed by David Hilbert inspired a great deal of progress in mathematics science [1]. Einstein was convinced that posing a new question and a new possibility to regard an old problem from a new angle need creative imagination and led to a real advance in science [2]. In 1980, Paul Halmos write down "I do believe that problems are the heart of mathematics". Both Cantor and Klamkin strongly believed that posing a problem is as important or more valuable than solving it [3]. Therefore, posing mathematical and statistical problems is more important than solving problems.

This paper is organized as follows: in Section 2, we give the definition of problem posing, and introduce the related research briefly. In Section 3, we put the model of problem-driven teaching, then give an application of the model in section 4. At last, give the discussion in Section 5 and the conclusions in Section 6.

2. Literature review

2.1. Definiation

Before the 1980s, mathematical problem-posing was not considered an independent activity but was seen as a part of problem-solving. Since the 1990s, problem posing has begun to be viewed as an independent mathematical activity. Silver [4] gave a classical definition of problem posing as the generation of new problems based on a given situation or the reformulation of given problems, which is generally accepted by the mathematics education community. Another definition of problem posing refers to several related types of activities that engage teachers and students in formulating (or reformulating) and expressing a problem according to a given situation [5]. The former definition focuses on the source of posing new problems; the latter definition tends to present problem posing as a teaching or learning activity.

2.2. Related research

Mathematical problem posing is positioned differently in different studies, such as focusing on problem posing as a cognitive activity, as a learning goal, or as an instructional approach [6].

Many national curriculum standards for mathematics include problem posing as a teaching goal. For instance, in China, the mathematics curriculum standards for senior high school advocate improving students' ability to discover problems, pose problems, analyze problems, and solve problems [7]. We already know that the curriculum standards place great emphasis on the importance of problem posing; that there is much research showing the positive effects of mathematical problem posing on students' interest in mathematics and creativity in mathematics as well as on self-efficacy and problem solving; and that there are many specific strategies or techniques for helping students to pose (better) mathematical problems, such as the strategies of "what-if-not" [8]; the strategies of chaining, systematic variation, and symmetry [9], and so on; There are also studies on the process, phases, or stages of problem posing, for example, Cruz's model of process (educative needs, goal, problem formulation, problem solving and problem improving) [10]; the five phases (setup, transformation, formulation, evaluation, and final assessment) [11]; another five types of activities (situation analysis, variation, generation, problem solving, evalution) [12]; and most recently, Cai's problem posing task-based instuctional model [13]. However, there is still a need to unfold the following directions: the design of problem-posing tasks in mathematics textbooks, teachers' design of problem-posing-based lessons, teachers' implementation of problem-posing in mathematics classrooms, and the assessment of problem-posing in teachers' teaching and students' learning, teachers' problem-posing beliefs and performance, students' problem-posing performance and influencing factors, and how problem posing promotes mathematics science.

3. The model of problem-driven teaching (PDT)

Today, mathematics teachers are progressively realizing that there are many benefits of PDT, but they also face many challenges, such as a lack of confidence, not knowing how to design high-quality problems, and how to deal with the challenges that emerge in problem-driven mathematics and statistics classes. Therefore, to answer these open questions, after dissecting the models (stages, phases, and processes) in the opening literature, combined with a delicate diagnosis of the first author's teaching experience, we aim to propose a ready-to-use model of PDT which concludes with three stages:

1. The stage of preparation of the problems (generating new problems based on textbooks; generating new problems based on mathematical, scientific, and life situations, imagining solutions of the prepared problems);
2. The stage of implementation of teaching (teachers teach based on prepared problems and pose new problems in real-time, and students solve them or pose new problems);
3. The evaluation and look-back stage (evaluating the quality of teaching and the quality of problems, improving the instruction).

We divided the problems into two categories: mathematical and non-mathematical problems, such as heuristic problems and prompts.

4. Application

Accordingly, we select the topic of "Estimating the population from a sample" in the General Senior Secondary Textbook, Mathematics, Compulsory Book 2 (People's Education Press, 2019) [14] to design a lesson/instruction that fulfills problems based on the model of PDT. Study how to design a frequency distribution table and frequency distribution histogram by a sample survey data and how to compute the average, percentile of the sample, therefore further go to estimate the numerical characteristics of the population.

4.1. The routine situation

The textbook [14] proposed the following problem situation:

To save water, a municipality plans to implement a tiered water pricing system for residential domestic water costs, i.e. to determine a standard α for the average monthly water consumption of residents, specifically, the portion of water consumption up to α will be charged at a flat rate and the portion above α will be charged at a negotiated rate. What do you need to do to try to determine a more reasonable standard so that the majority of residential customers are not affected by their water fees? Because of the large number of residential customers in the city, a sample survey is generally used to estimate the distribution of average monthly water consumption of residential customers in the city by analyzing sample observations. Therefore, the overall population is all residential customers in the city, the individual is each residential customer and the variable surveyed is the average monthly water consumption of residential customers. By a random sample survey, a data table of the average monthly water consumption of 100 residential customers (in t) was obtained, see Table A1.

4.2. The first stage

In the stage of preparing problems, the teacher needs to understand the given situation, identify information and discover or create problems about quantity, space, or logic that may be therein; generate new problems based on the given situation; the teacher gives different possible mathematical, scientific and life situations; generate new problems based on mathematical, scientific and life situations and imagine solutions to the prepared problems.

4.3. The second stage

In the stage of implementation of teaching, the place where it happens should be in the classroom. According to the presetting in the first stage, teachers should teach based on prepared problems and pose new problems in real-time, and students solve them or pose new problems based on the problem situation envisaged by the students themselves. The teacher may give appropriate prompts for the problem-posing task or give good example problems so that students can trigger problem-posing and formulate their problems correctly and positively. Here the problem posed by teachers and students are not linear and can be disorder and crossed. To enable students to pose effective problems, teachers should introduce some strategies, such as "what-if-not", chaining, systematic variation, symmetry, analogies, associations, considering the counter-problems, extensions, reductions, specializations (considering special cases), generalization, varying the givens, imitation. To record the lessons, information technology such as video and audio recordings can be used. Students are also asked to fill in the notebook with their posed problems, and the teacher will collect them for analysis and evaluation, and improvement after the bell rings.

4.4. The third stage

There are three different aspects to the evaluation in the evaluation and look-back stage: (1) the evaluation of the teacher's problem posing; (2) the evaluation of the student's problem–posing; (3) the evaluation of the quality of the teacher's teaching. The evaluation of the quality of the posed problem from fluency, flexibility, and profundity [15].

4.5. Possible problems

We demonstrate the possible problems that teachers and students can blurt out.

1. If the α is too large or too small, what are the effects? Why?
2. There are many households, how do you survey them?

3. What questions can be asked about the data (Table A1)? (Teachers introduce the frequency distribution table and the frequency distribution histogram)
4. Teacher introduces the concept of range and then poses the problem:
   If you modify (add or delete) some of the data in the table, but do not change the maximum and minimum values, find the range.
5. What should be a reasonable number of groups? Can the group size be very large or very small?
6. Why is the interval closed on the left and open on the right in Figure A1, except for the last one?
7. Why is it necessary to group them equidistantly? what if not?
8. Does change just some of the data change the number of groups?
9. What kinds of problems can be posed when looking at Figure A1? For instance, why sum of frequencies equals 1? Even if we modify (add or delete) some data, how about the sum?
10. What kinds of problems can be posed when looking at Figure A2? For instance, calculate the sum of the areas of all the rectangles.
11. If part of the data has been modified (added or deleted), calculate the sum of the areas of all the rectangles.
12. Why is it a dromedary? Why are the rectangles on the right side very short?
13. Calculate the plurality of the sample and compare with Figure A1.
14. Calculate the median of the sample.
15. Calculating the quartiles of the sample.
16. Calculating the three-quarters of the sample.
17. Calculating the mean of the sample.
18. If the data are divided into 10 equally spaced groups, design the frequency distribution table, and the histogram of the fractional distribution, find the plural, median and mean.
19. Design new problem situations and pose the corresponding data and problems.
20. ...

5. Discussion

More investigation is needed to examine whether students' mathematics performance, ability to pose problems, interest in learning mathematics, and creativity have improved after they experience PDT. Later studies may establish a more scientific and rational system of evaluation indexes. The assessment of teaching quality may take more into account students' attitudes to learning, learning outcomes, and academic performance, which is still being conceptualized.

6. Conclusions

In the lesson/instruction, several interesting problems were posed by teachers or students. These problems can successfully stimulate students' interest in learning statistics. Through these problems, the maths teacher can guide the students to think actively and pose and solve the problem themselves, thus understanding the related concepts and methods in statistics. In addition, the design model applied here is a ready-to-use tool for teachers in designing specifically problem-driven instruction and also can improve teachers' problem-posing skills.

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Appendix A

The appendix is an optional section that can contain details and data supplemental to the main text—for example, explanations of experimental details that would disrupt the flow of the main text but nonetheless remain crucial to understanding and reproducing the research shown; figures of replicates for experiments of which representative data are shown in the main text can be added here if brief, or as Supplementary Data. Mathematical proofs of results not central to the paper can be added as an appendix.

Table A1. A random sample data table of the average monthly water consumption of 100 residential customers (in t) (cf. [14], pp. 192–193).

9.0	13.6	14.9	5.9	4.0	7.1	6.4	5.4	19.4	2.0
2.2	8. 6	13.8	5.4	10.2	4.9	6.8	14.0	2.0	10.5
2.1	5.7	5.1	16.8	6.0	11.1	1.3	11.2	7.7	4.9
2.3	10.0	16.7	12.0	12.4	7.8	5.2	13.6	2.6	22.4
3.6	7.1	8.8	25.6	3.2	18.3	5.1	2.0	3.0	12.0
22.2	10.8	5.5	2.0	24.3	9.9	3.6	5.6	4.4	7.9
5.1	24.5	6.4	7.5	4.7	20.5	5.5	15.7	2.6	5.7
5.5	6.0	16.0	2.4	9.5	3.7	17.0	3.8	4.1	2.3
5.3	7.8	8.1	4.3	13.3	6.8	1.3	7.0	4.9	1.8
7.1	28.0	10.2	13.8	17.9	10.1	5.5	4.6	3.2	21.6

分组	频数累计	频数	频率
[1.2，4.2)	正正正正下	23	0. 23
[4.2，7.2)	正正正正正正丅	32	0. 32
[7.2，10.2)	正正下	13	0. 13
[10.2，13.2)	正𤴓	9	0. 09
[13.2，16.2)	正𤴓	9	0. 09
[16.2，19.2)	正	5	0. 05
[19.2，22.2)	下	3	0. 03
[22.2，25.2)	𤴓	4	0. 04
[25.2，28.2]	丅	2	0. 02
合计		100	1. 00

Figure A1. Frequency distribution table (cf. [14], pp. 194).

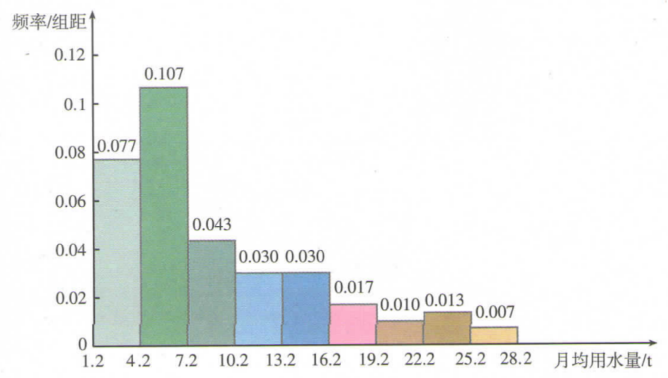

Figure A2. Frequency distribution histogram (cf. [14], pp. 195).

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