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Optimal Block Replacement Policies under Replacement First and Last Disciplines

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Abstract: This paper considers opportunity-based block replacement problems, where the inter-arrival times of replacement opportunities arrive randomly. Specifically, we first propose the opportunity-based block replacement models under the replacement first and last disciplines in the sense of Zhao and Nakagawa (2012). Then, we derive the optimal replacement policies under the replacement first and last disciples. Finally, numerical examples are presented to compare these optimal replacement policies.

Keywords: Opportunity-based block replacement; Expected cost functions; Replacement first/last disciplines.

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1. Introduction

Various simple but realistic preventive maintenance (PM) policies have been extensively studied to maintain system dependability. PM is generally a set of activities, including repair maintenance, age replacement, periodic replacement, and block replacement [1]. These activities can be performed before a system failure to reduce the maintenance cost at failure time and improve system reliability. Among these replacement policies, block replacement is a common method for maintaining the reliability of systems that consist of a block or group of units. However, as systems become more complex and stochastic, a random maintenance policy would fit more for the systems and should be done rather than a deterministic one [2]. Several random block replacement models have been proposed. For example, Sheu [3] presented a block replacement policy with used items and considered general random minimal repair cost, and he further generalized the replacement by taking account of random shocks [4]. Anisimov [5] analyzed asymptotic properties of stochastic block replacement policies in a Markov environment. In [6], Sheu et al. reformulated the block replacement models in which the occurrence of shocks obeys a non-homogeneous pure birth process. Yao and Zhou [7] proposed a new type of uncertain random process for block replacement, called the uncertain random renewal reward process, where the inter-arrival times and the rewards were assumed to be random variables and uncertain variables, respectively. Park and Pham [8] elaborated on cost models by examining the renewable and non-renewable warranty policies subject to minimal repair within the warranty period and the post-warranty period. Recently, Sheu et al. [9] discussed a preventive replacement policy with the concept of replacement last under cumulative damage models, and determined the optimal policies that minimize the average cost rate.

On the other hand, opportunity plays an important role in preventive replacement. The opportunity is a chance to make the replacement, and one can replace a unit with a low cost at the opportunity. Dekker and Smeitink [10] considered a block replacement model in which replacement can be replaced preventively at maintenance opportunities. Cavalcante et al. [11] proposed an inspection and opportunistic replacement policy for one-unit systems.

Zhao and Nakagawa [12], Zhao et al. [13], and Zhao et al. [14] formulated somewhat different opportunity-based models, called replacement first (RF) and replacement last (RL). Essentially, RF and RL disciplines mix standard and random age replacement. In context, the RF discipline is described as a preventive replacement that is performed at an opportunity arrival time or a pre-scheduled replacement time, whichever occurs first. Conversely, the RL discipline is that a preventive replacement is performed at an arrival time of opportunity or a pre-scheduled replacement time, whichever occurs last [15]. In this paper, we formulate the block replacement models under RF and RL disciplines in sense of Zhao and Nakagawa [6]. Moreover, we extend the minimal repair model, a special case of block replacement models, where preventive replacement cost is not equal to opportunistic replacement cost.

The remainder of this paper is organized as follows. Section 2 describes the models and derives the average cost functions. In Section 3, a numerical illustration is presented to compare these optimal policies under the RF and RL disciplines. Section 4 concludes the paper with directions for future work.

2. Model Description

2.1. Notation and Assumption

Consider a unit that undertakes the preventive replacement at the pre-scheduled replacement time or an opportunity. Replacements are assumed to occur instantaneously. The lifetime of the unit is represented by the continuous random variable (r.v.) X. Let $F(t)$, $f(t)$, $M(t)$, and $m(t)$ denote the corresponding cumulative distribution function (c.d.f.), probability density function (p.d.f.), renewal function, and renewal density function, respectively. Opportunities occur according to a renewal process, independently of the lifetime process. Without loss of generality, the failure rate function is given by $h(t) = f(t)/\overline{F}(t)$. Let the continuous r.v. Y denote the inter-arrival time between two consecutive opportunities, and $G(t)$ and $g(t)$ represent its corresponding c.d.f. and p.d.f., respectively. Besides, we denote the following replacement costs:

- c_F: The failure replacement cost.
- c_P: The preventive replacement cost.
- c_Y: The opportunistic replacement cost.
- c_M $(< c_F)$: The minimal repair cost at a failure.

Note that it is generally supposed that $c_F > c_P \geq c_Y$. It would be valid to assume that the corrective replacement cost, c_F, is the most expensive. The opportunistic replacement cost, c_P, is less than that of the preventive replacement, because the opportunity means a chance to make the replacement, and one can replace a unit with a low cost at the opportunity. Osaki [16] summarized several basic models of block replacement policies. In this paper, we formulate two more common models among these models; that is, block replacement models and minimal repair models, under RF and RL disciplines.

2.2. Block Replacement Model

2.2.1. Block Replacement Model with RF

Suppose that the unit is replaced at a total operating time T $(0 < T < \infty)$ or at a random working cycle Y, whichever occurs first, and undergoes failure repair at each failure between replacements. Let $C(T)$ denote the long-run average cost in the steady state. Then we have

$$C(T) = \frac{B(T)}{A(T)}, \tag{1}$$

where $A(T)$ and $B(T)$ are the expected cycle length and the expected cost per cycle, respectively. Note that a cycle corresponds to the time length between consecutive renewal points, including preventive replacement, opportunistic replacement, and failure replacement. It is straightforward to derive $A(T)$ and $B(T)$ as

$$A(T) = \int_0^T \overline{G}(t)dt, \tag{2}$$

$$\begin{aligned} B(T) &= c_F \int_0^T \overline{G}(t)m(t)dt + c_Y G(T) + c_P \overline{G}(T) \\ &= c_P + c_F \int_0^T \overline{G}(t)m(t)dt - (c_P - c_Y)G(T). \end{aligned} \tag{3}$$

Our purpose is to derive the optimal preventive time T^* minimizing the long-run average cost $C(T)$. Thus, differentiating $C(T)$ with respect to T and setting it equal to zero yield

$$c_F \int_0^T \overline{G}(t)[m(T) - m(t)]dt - (c_P - c_Y) \int_0^T \overline{G}(t)[r(T) - r(t)]dt = c_P, \tag{4}$$

where $r(t) = g(t)/\overline{G}(t)$ is called the hazard rate function. Denoting $Q_1(T)$ as the left-hand of Eq. (4), we have

$$Q_1(0) = 0 < c_P. \tag{5}$$

Theorem 1. *Suppose that $m(t)$ is increasing in t and $h(t)$ is decreasing in t.*

1. *If $Q_1(\infty) \geq c_P$, then there exists a finite and unique T^* $(0 < T^* < \infty)$ and the resulting cost rate is*

$$C(T^*) = c_F m(T^*) - (c_P - c_Y)r(T^*). \tag{6}$$

2. *If $Q_1(\infty) < c_P$, then the optimal replacement time is given by $T^* \to \infty$.*

2.2.2. Block Replacement Model with *RL*

For the model under RL discipline, the expected cycle length $A(T)$ and the expected cost per cycle $B(T)$ are given by

$$A(T) = TG(T) + \int_T^\infty tg(t)dt = T + \int_T^\infty \bar{G}(t)dt, \tag{7}$$

$$\begin{aligned} B(T) &= c_F \left(M(t)G(t) + \int_T^\infty m(t)\overline{G}(t)dt \right) + c_P G(T) + c_Y \overline{G}(T) \\ &= c_P + c_F \left(M(T) + \int_T^\infty \overline{G}(t)m(t)dt \right) - (c_P - c_Y)\overline{G}(T), \end{aligned} \tag{8}$$

respectively. After substituting Eqs. (7) and (8) into Eq. (1), the long-run average cost function $C(T)$ is obtained. We then differentiate $C(T)$ with respect to T and set it equal to zero, and finally have

$$\begin{aligned} & c_F \left[\int_0^T [m(T) - m(t)]dt + \int_T^\infty \overline{G}(t)[m(T) - m(t)]dt \right] \\ & + (c_P - c_Y) \left[\hat{r}(T)T + \hat{r}(T) \int_T^\infty \overline{G}(t)dt + \overline{G}(t) \right] = c_P, \end{aligned} \tag{9}$$

where $\hat{r}(t) = g(t)/G(t)$ is called the revised hazard rate function. Let $Q_2(T)$ denote the left-hand of the above equation.

Theorem 2. *Suppose that $c_F m(T) + (c_P - c_Y)\hat{r}(T)$ is increasing in T.*

1. *If* $Q_2(0) < c_P < Q_2(\infty)$ *then there exists a finite and unique* T^* $(0 < T^* < \infty)$ *and the resulting cost rate is*

$$C(T^*) = c_F m(T^*) + (c_P - c_Y)\hat{r}(T^*). \tag{10}$$

2. *If* $Q_2(0) \geq c_P$, *then the optimal replacement time is* $T^* = 0$.
3. *If* $Q_2(\infty) \leq c_P$, *then the optimal replacement time becomes* $T^* \to \infty$.

2.3. Minimal Repair Model

Zhao and Nakagawa [7] supposed that the cost of preventive replacement equals to the cost of opportunistic replacement. Obviously, it is not realistic. In this section, we reformulate the minimal repair model under RF and RL disciplines [9] where the cost of preventive replacement is not equal to the cost of opportunistic replacement. Similar to the block replacement models described in Sect. 2.2, we introduce the minimal repair models under both RF and RL disciplines as follows.

2.3.1. Minimal Repair Model with RF

Suppose that the unit is replaced at a total operating time T $(0 < T \leq \infty)$ or at opportunities Y, whichever occurs first, and undergoes minimal repair at each failure between replacements. The expected cost rate $C(T)$ is given by

$$C(T) = \frac{c_P + c_M \int_0^T \overline{G}(t)h(t)dt - (c_P - c_Y)G(T)}{\int_0^T \overline{G}(t)dt}. \tag{11}$$

Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$c_M \int_0^T \overline{G}(t)[h(T) - h(t)]dt - (c_P - c_Y)\int_0^T \overline{G}(t)[r(T) - r(t)]dt = c_P. \tag{12}$$

Let $Q_3(T)$ denote the left-hand of the above equation, that is

$$Q_3(0) = 0 < c_P. \tag{13}$$

Theorem 3. *Suppose that* $h(t)$ *is increasing in* t *and* $r(t)$ *is decreasing in* t.

1. *If* $Q_3(\infty) \geq c_P$, *then there exists a finite and unique* T^* $(0 < T^* < \infty)$ *and the resulting cost rate is*

$$C(T^*) = c_M h(T^*) - (c_P - c_Y)r(T^*). \tag{14}$$

2. *If* $Q_3(\infty) < c_P$, *then the optimal replacement time is* $T^* \to \infty$.

2.3.2. Minimal Repair Model with RL

Suppose that the unit is replaced at a total operating time T $(0 < T \leq \infty)$ or at opportunities Y, whichever occurs last. The expected cost in the long-run time $C(T)$ under RL discipline is given by

$$C(T) = \frac{c_P + c_M\left[H(T) + \int_T^\infty \overline{G}(t)h(t)dt\right] - (c_P - c_Y)\overline{G}(T)}{T + \int_T^\infty \overline{G}(t)dt}, \tag{15}$$

where $H(t) = \int_0^t h(u)du$.

Table 1. Comparison of block replacement models under RF and RL ($\lambda = 1, \sigma = 2, c_F = 200$).

	$c_Y = 10$				$c_Y = 5$			
	RF		RL		RF		RL	
c_P	T^*	$C(T^*)$	T^*	$C(T^*)$	T^*	$C(T^*)$	T^*	$C(T^*)$
10	0.5005	63.25	0.5034	63.46	0.5005	53.25	0.3986	63.14
12	0.5902	69.28	0.5442	66.32	0.5902	59.28	0.5203	66.02
14	0.6898	74.83	0.5868	69.07	0.6898	64.83	0.6945	68.79
16	0.8047	80.00	0.6315	71.72	0.8047	70.00	0.7933	71.45
18	0.9437	84.85	0.6784	74.25	0.9437	74.85	0.9607	74.02
20	1.1242	89.44	0.7278	76.67	1.1242	79.44	1.1164	76.47
22	1.3910	93.81	0.7800	78.98	1.3910	83.81	1.4426	78.81
24	1.9509	97.98	0.8355	81.19	1.9509	87.98	1.8901	81.04
26	∞	200	0.8946	83.29	∞	200	2.4520	83.16

Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$c_M\left[\int_0^T [h(T)-h(t)]dt + \int_T^\infty \overline{G}(t)[h(T)-h(t)]dt\right] + (c_P - c_Y)\left[\hat{r}(T)T + \hat{r}(T)\int_T^\infty \overline{G}(t)dt + \overline{G}(t)\right] = c_P. \tag{16}$$

Let $Q_4(T)$ denote the left-hand of the above equation.

Theorem 4. *Suppose that $c_M h(T) + (c_P - c_Y)\hat{r}(T)$ is increasing in T.*

1. *If $Q_4(0) < c_P < Q_4(\infty)$, then there exists a finite and unique T^* $(0 < T^* < \infty)$ and its resulting cost rate is*
$$C(T^*) = c_M h(T^*) + (c_P - c_Y)\hat{r}(T^*). \tag{17}$$
2. *If $Q_4(0) \geq c_P$, then the optimal replacement time is $T^* = 0$.*
3. *If $Q_4(\infty) \leq c_P$, then the optimal replacement time becomes $T^* \to \infty$.*

3. Numerical Examples

Suppose that the lifetime time X of an unit follows a Gamma distribution with the p.d.f. $f(t) = [\lambda(\lambda t)^{\alpha-1}/\Gamma(\alpha)]e^{-\lambda t}$ in which $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, $0 < x < \infty$ and $\alpha = 2$, i.e., $F(t) = 1 - (1+\lambda)e^{-\lambda t}$ [2]. Then, we have

$$m(t) = \frac{1}{2}\lambda\left(1 - e^{-2\lambda t}\right), \quad M(t) = \frac{1}{2}\lambda t + \frac{1}{4}e^{-2\lambda t} - \frac{1}{4}. \tag{18}$$

Furthermore, $G(t) = 1 - e^{-\sigma t}$.

Table 1 presents the optimal preventive time T^* and the minimum expected cost $C(T^*)$ with the optimal preventive time for block replacement with $c_Y = 5$ and 10. It is clear that when c_P/c_F is small, the model under RF is better than the one under RL. In particular, when the value of c_Y is small, the tendency becomes more significant.

The optimal preventive time T^* and the corresponding minimum expected cost $C(T^)*$ for minimal repair with $c_Y = 5$ and 10 are demonstrated in Table 2. From the table, we see that when $c_P = c_Y$, the model under RL outperforms the one under RF. When $c_P > c_Y$, if c_P/c_F is small, the replacement first is better than the replacement last. Conversely, if c_P/c_F is big enough, the replacement last is better than the replacement first.

Table 2. Comparison of minimal repair models under RF and RL ($\lambda = 1$, $\sigma = 2$, $c_M = 100$).

	$c_Y = 10$				$c_Y = 5$			
	RF		RL		RF		RL	
c_P	T^*	$C(T^*)$	T^*	$C(T^*)$	T^*	$C(T^*)$	T^*	$C(T^*)$
10	0.8485	45.29	0.6837	40.82	0.8485	35.29	0.5498	40.43
15	1.3408	57.27	0.8624	46.65	1.3408	47.27	0.7679	46.17
20	2.0934	67.69	1.0480	51.33	2.0934	57.69	0.9778	51.08
25	3.4908	77.73	1.2404	55.36	3.4908	67.73	1.1875	55.31
30	7.1528	87.75	1.4396	59.19	7.1528	77.75	1.3997	58.97
35	43.15	97.74	1.6461	62.21	43.15	87.74	1.6162	62.12
40	∞	100	1.8604	65.04	∞	100	1.8383	65.00
45	∞	100	2.0832	67.57	∞	100	2.0672	67.55
50	∞	100	2.3153	69.85	∞	100	2.3038	69.81

4. Concluding Remarks

In this paper, we presented opportunity-based block replacement models under replacement first and last disciplines. We characterized the uniqueness of the optimal scheduled preventive replacement times, which minimize the expected costs per unit time in the steady state. In the numerical example, we compared two replacement policies, replacement first policies and replacement last policies. In block replacement models, replacement first is better than replacement last in some limited cases where c_P/c_F is small. In minimal repair models, if c_P/c_F is big enough, the replacement last is better than the replacement first.

In future work, we will generalize opportunity-based block replacement models using Markov arrival process, since as pointed out by Zheng et al. [10], it is more realistic to consider the case where the arrivals of opportunities obey a Markov process.

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