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The Asymptotics of Moments for the Remaining Time of the Heavy-Tail Distributions †

Vladimir Rusev [1,*,‡] and Alexander Skorikov [2,*,‡]

National University of Oil and Gas (Gubkin University), Moscow, Russian
* Correspondence: rusev.v@gubkin.ru (V.R.); skorikov.a@gubkin.ru (A.S)
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‡ These authors contributed equally to this work.

Abstract: Recent mathematical models of reliability computer systems and telecommunication networks are based on distributions with heavy tails. This paper falls into category exploring the classical models with heavy tails: Gnedenko-Weibull, Burr, Benktander distributions. The moment's asymptotics for residual time have been derived especially for this heavy-tailed distributions. The simulation study is presented to give a practice characterization distributions with heavy tails.

Keywords: heavy-tail distributions; asymptotic expansion; mean residual life function; mean excess functions; residual variance

1. Motivation and Background

Definition 1. *One defines a cumulative distribution function (DF) F to be (right-) heavy-tailed if and only if*

$$\lim_{x\to+\infty} e^{\lambda x}\overline{F}(x) = \infty$$

for the tail: $\overline{F} = 1 - F$ *and for all* $\lambda > 0$ *.*

Thus, one shall say that tail of distribution (tending to zero) is heavy-tailed if it fails to be bounded by a decreasing exponential function.

Definition 2. *A cumulative distribution function F is called light-tailed if and only if it fails to be heavy-tailed.*

The slow decrease the tail of distribution leads to the fact that random variable can take on really large values with a positive probability. So such distributions are used to model phenomena that are subject to strong fluctuations. Thus, the sample contains mainly relatively small values, but also has a sufficient number of very large values as illustrated by the plot in Figure 1 for 200 realisations of independent, identically distributed random variables for heavy-tailed (Gnedenko-Weibull DF with less then 1 shape parameter) and light-tailed (exponential) DF.

Innovated mathematical models of the computer systems and telecommunications networks well characterized by heavy-tailed distributions. For example the superposition of ON/OFF processes with a heavy-tailed distribution was presented in [3]. Such processes converges to a self-similar (fractal) process with self-similarity Hurst exponent $H = (3-\alpha)/2$, where $0 < \alpha < 2$ is the tail index (or coefficient of heavy-tailedness).

It is well known a correspondence between the generated traffic and the actual Ethernet traffic. Empirical self-similar processes of Ethernet traffic remains unchanged at varying time scales unlike Poisson processes, which flatten out as time scales change.

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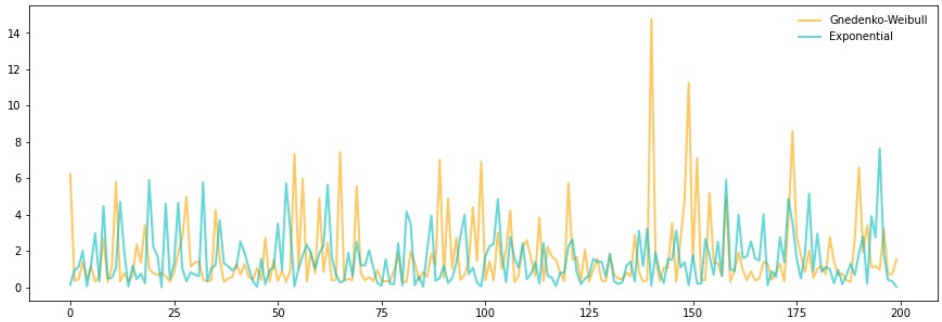

Figure 1. Synthetic plot generated from comparable composite samples of Gnedenko-Weibull distribution with shape parameter 0.69 and scale parameter 1 and exponential distribution with parameter 0.64: (**a**) Gnedenko-Weibull DF, (**b**) exponential DF.

Let X be non-negative random variable with cumulative distribution function F. Consider the random variable $X_t = (X - t|X > t)$, which is called in survival analysis the residual life time with distribution function F_t or distribution of the excess over a threshold t. The function

$$\bar{F}_t(x) = 1 - F_t(x) = P(X - t > x|X > t)$$

is known as distribution tail. The mathematical expectation of random variable X_t is the mean residual life (MRL) or mean excess function (ME). The ME term is using in risk management, actuarial science and other extreme value problems. This function is defined as

$$\mu(t) = E(X - t|X > t) = \int_t^{+\infty} \overline{F}(x)dx/\overline{F}(t) \tag{1}$$

and residual variance (excess variance)

$$\sigma^2(t) = E(X_t^2) - \mu^2(t) = \frac{2}{\overline{F}(t)} \int_t^{+\infty} \overline{F}(x) \cdot \mu(x)\, dx - \mu^2(t). \tag{2}$$

The physical application of MRL to reliability of the "well - pump" system has been reported in [4].

On the other hand, the accuracy of the simulation obtained depends on the accuracy of the approximation of the distributions with a heavy tail used. In model selection it is advisable to study residual lifetime in terms of its moments for relatively small t. A more precise asymptotic expansion moments this to be done. We have obtained the new asymptotic expansion terms of the mean excess function (mean residual) and residual variance asymptotic expansion for known distributions.

The paper is organized as follows. In Section 2, we consider the problem of asymptotics of the residual lifetime distribution (mean excess function) for the classical models (Tables 1.25, 1.2.6 [1]). We derive some refinements of asymptotic expansions from (Table 3.4.7 in [1], p. 161). Asymptotic expansions for the residual variance are obtained too.

2. Residual Moments for Heavy Tails

2.1. Gnedenko-Weibull Distribution

The tail two-parameter Gnedenko-Weibull distribution

$$\bar{F}(t;\alpha,\beta) = e^{-(\alpha t)^{\beta}}, \quad t \geq 0$$

is a heavy tail if $0 < \alpha, \quad 0 < \beta < 1$.

Let T be a random variable with the two-parameter Gnedenko-Weibull distribution, then $\forall\alpha > 0,\ \forall\beta > 0$ holds

$$\mu(t) = \frac{t}{\beta(\alpha t)^{\beta}}\left(1 + \frac{1-\beta}{\beta(\alpha t)^{\beta}} + \frac{(1-\beta)(1-2\beta)}{\beta^2(\alpha t)^{2\beta}} + O\left(\frac{1}{t^{3\beta}}\right)\right), \quad \text{as } t \to \infty, \tag{3}$$

$$\sigma^2(t) = \frac{1}{\beta^2\alpha^{2\beta}t^{2(\beta-1)}}\left(1 + \frac{4(1-\beta)}{\beta(\alpha t)^{\beta}} + \frac{(1-\beta)(11-17\beta)}{\beta^2(\alpha t)^{2\beta}} + O\left(\frac{1}{t^{3\beta}}\right)\right), \text{as } t \to \infty. \tag{4}$$

The proof of asymptotic expressions for the mean residual life and residual variance is considered in [2].

2.2. Benktander Type I Distribution

Benktander type I distribution is close to lognormal distribution. This distributions are using to model heavy tailed losses in actuarial science. Theoretically predicted sizes by Benktander I distribution are consistent with statistical size distributions in economics and actuarial sciences. The Benktander type I distribution function tail is given by

$$1 - F(x) = \left(1 + 2\frac{\beta}{\alpha}\ln x\right)e^{-\beta(\ln x)^2-(\alpha+1)\ln x}, \quad \alpha > 0,\ \beta > 0,\ x > 0.$$

It then follows that

$$\int_t^{\infty}(1-F(x))dx = \int_t^{\infty}\left(1 + 2\frac{\beta}{\alpha}\ln x\right)e^{-\beta(\ln x)^2-(\alpha+1)\ln x}dx =$$

$$= \int_{\ln t}^{\infty}\left(1 + 2\frac{\beta}{\alpha}z\right)e^{-\beta(z)^2-(\alpha+1)z}e^z dz = \int_{\ln t}^{\infty}\left(1 + 2\frac{\beta}{\alpha}z\right)e^{-\beta z^2-\alpha z}dz =$$

$$= \int_{\ln t}^{\infty}\left(1 + 2\frac{\beta}{\alpha}z\right)e^{-\beta\left(z^2+\frac{\alpha}{\beta}z\right)}dz.$$

Let be $\gamma = \frac{\beta}{\alpha}$. Mean excess function is directly calculate:

$$\int_{\ln t}^{\infty}(1+2\gamma z)e^{-\beta\left(z^2+\gamma z\right)}dz = \int_{\ln t}^{\infty}(1+2\gamma z)e^{-\frac{\beta}{\gamma}\left(\gamma z^2+z\right)}dz =$$

$$= \int_{\ln t}^{\infty}e^{-\frac{\beta}{\gamma}\left(\gamma z^2+z\right)}d\left(\gamma z^2+z\right) = -\frac{\gamma}{\beta}\int_{\ln t}^{\infty}e^{-\frac{\beta}{\gamma}\left(\gamma z^2+z\right)}d\left(-\frac{\beta}{\gamma}\right)\left(\gamma z^2+z\right) =$$

$$= -\frac{\gamma}{\beta}e^{-\frac{\beta}{\gamma}\left(\gamma z^2+z\right)}\Big|_{\ln t}^{\infty} = \frac{\gamma}{\beta}e^{-\frac{\beta}{\gamma}\left(\gamma(\ln t)^2+\ln t\right)} = \frac{1}{\alpha}e^{-\alpha\left(\frac{\beta}{\alpha}(\ln t)^2+\ln t\right)};$$

$$\mu(t) = \frac{\int_t^{+\infty}(1-F(x))dx}{1-F(t)} = \frac{\frac{1}{\alpha}e^{-\alpha\left(\frac{\beta}{\alpha}(\ln t)^2+\ln t\right)}}{\left(1+2\frac{\beta}{\alpha}\ln t\right)e^{-\beta(\ln t)^2-(\alpha+1)\ln t}} =$$

$$= \frac{e^{-\beta(\ln t)^2}t^{-\alpha}}{(\alpha+2\beta\ln t)e^{-\beta(\ln t)^2}t^{-\alpha-1}} = \frac{t}{\alpha+2\beta\ln t}.$$

Thus, the Type I Benkthander distribution has the following exact equality for the mean exceedance:

$$\mu(t) = \frac{t}{\alpha + 2\beta \ln t}. \tag{5}$$

Note that (5) is contained in the (Table 3.4.7, [1], p.161) without proof. The proof is given for fullness.

From the above result and the presentation for the integral

$$\int_x^{\infty} e^{-z^2} dz = e^{-x^2}\left(\frac{1}{2x} + \frac{1}{4x^3} + O\left(\frac{1}{x^5}\right)\right), \quad \text{as } x \to \infty$$

it follows

$$\int_t^{+\infty} (1 - F(x)) \cdot \mu(x)\, dx = \frac{1}{\alpha} \int_t^{+\infty} e^{-\beta(\ln x)^2 - (\alpha+1)\ln x} x dx =$$

$$= \frac{1}{\alpha} \int_{\ln t}^{\infty} e^{-\beta(z)^2 - (\alpha+1)z} e^{2z} dz = \frac{1}{\alpha} \int_{\ln t}^{\infty} e^{-\beta(z)^2 - \alpha z + z} dz =$$

$$= \frac{e^{\frac{(\alpha-1)^2}{4\beta}}}{\alpha} \int_{\ln t}^{\infty} e^{-\beta\left(z + \frac{\alpha-1}{2\beta}\right)^2} d\left(z + \frac{\alpha-1}{2\beta}\right) = \frac{e^{\frac{(\alpha-1)^2}{4\beta}}}{\alpha\sqrt{\beta}} \int_{\sqrt{\beta}\left(\ln t + \frac{\alpha-1}{2\beta}\right)}^{\infty} e^{-u^2} du =$$

$$= \frac{e^{\frac{(\alpha-1)^2}{4\beta}}}{\alpha\sqrt{\beta}} e^{-\beta\left(\ln t + \frac{\alpha-1}{2\beta}\right)^2} \left[\frac{1}{2\sqrt{\beta}\left(\ln t + \frac{\alpha-1}{2\beta}\right)} + \frac{1}{4\left(\sqrt{\beta}\left(\ln t + \frac{\alpha-1}{2\beta}\right)\right)^3} + \cdots\right].$$

Residual variance according as (2) and above asymptotics is derives by equation

$$\sigma^2(t) = \frac{2\alpha}{\alpha + 2\beta \ln t} \cdot e^{\beta(\ln t)^2 + (\alpha+1)\ln t} \cdot \frac{e^{\frac{(\alpha-1)^2}{4\beta}}}{\alpha\sqrt{\beta}} e^{-\beta\left(\ln t + \frac{\alpha-1}{2\beta}\right)^2} \times$$

$$\times \left[\frac{1}{2\sqrt{\beta}\left(\ln t + \frac{\alpha-1}{2\beta}\right)} + \frac{1}{4\left(\sqrt{\beta}\left(\ln t + \frac{\alpha-1}{2\beta}\right)\right)^3} + O\left(\frac{1}{(\ln t)^5}\right)\right] - \frac{t^2}{(\alpha + 2\beta \ln t)^2} =$$

$$= \frac{2t^2}{\alpha + 2\beta \ln t} \cdot \left[\frac{1}{(2\beta \ln t + \alpha - 1)} + \frac{1}{4\beta\left(\ln t + \frac{\alpha-1}{2\beta}\right)^3} + O\left(\frac{1}{(\ln t)^5}\right)\right] - \frac{t^2}{(\alpha + 2\beta \ln t)^2} =$$

$$= \frac{t^2}{\alpha + 2\beta \ln t}\left[\frac{2}{(2\beta \ln t + \alpha - 1)} - \frac{1}{(2\beta \ln t + \alpha)} + \frac{4\beta^2}{(2\beta \ln t + \alpha - 1)^3} + O\left(\frac{1}{(\ln t)^5}\right)\right] =$$

$$= \frac{t^2}{(\alpha + 2\beta \ln t)^2}\left[\frac{2\beta \ln t + \alpha + 1}{(2\beta \ln t + \alpha - 1)(2\beta \ln t + \alpha)} + O\left(\frac{1}{(\ln t)^3}\right)\right] =$$

$$= \frac{t^2}{(\alpha + 2\beta \ln t)^2}\left[1 + \frac{2}{2\beta \ln t + \alpha - 1} + O\left(\frac{1}{(\ln t)^3}\right)\right] \quad \text{as } t \to \infty.$$

Hence

$$\sigma^2(t) = \frac{t^2}{(\alpha + 2\beta \ln t)^2}\left[1 + \frac{2}{2\beta \ln t + \alpha - 1} + O\left(\frac{1}{(\ln t)^3}\right)\right], \text{as } t \to \infty. \tag{6}$$

By binomial expansion formula, one derives residual standard deviation

$$\sigma(t) = \frac{t}{\alpha + 2\beta \ln t}\left[1 + \frac{1}{2\beta \ln t + \alpha - 1} - \frac{1}{4} \cdot \frac{1}{(2\beta \ln t + \alpha - 1)^2} + o\left(\frac{1}{(\ln t)^2}\right)\right], \quad (7)$$

$$\text{as } t \to \infty.$$

2.3. Benktander Type II Distribution

This distribution is near to the Gnedenko-Weibull. The formula Benktander type II average residual resource looks like the asymptotics of the average residual resource of the Weibull distribution. But it is an exact equality, yet for the residual variance there is only an asymptotic expansion. Consider the Benktander type II distribution with distribution tail

$$1 - F(t) = e^{\frac{\alpha}{\beta}} t^{\beta-1} e^{-\frac{\alpha}{\beta}t^\beta} \ \alpha > 0,\ 0 < \beta < 1. \quad (8)$$

It is immediately follows from (1), see Table 3.4.7 , [1]

$$\mu(t) = \frac{1}{\alpha} t^{1-\beta},\ \alpha > 0,\ 0 < \beta < 1. \quad (9)$$

Using the presentation (9) of $\mu(t)$ the following formula is easily derived

$$\int_t^{+\infty} (1 - F(x)) \cdot \mu(x)\, dx = \frac{e^{\frac{\alpha}{\beta}}}{\alpha} \int_t^{+\infty} e^{-\frac{\alpha}{\beta}x^\beta} dx.$$

Substituting $z = \frac{\alpha}{\beta}x^\beta$ into integral above we obtain

$$\int_t^{+\infty} e^{-\frac{\alpha}{\beta}x^\beta} dx = \frac{\beta^{\frac{1}{\beta}-1}}{\alpha^{\frac{1}{\beta}}} \int_{\frac{\alpha}{\beta}t^\beta}^{+\infty} e^{-z} z^{\frac{1}{\beta}-1} dz = \frac{\beta^{\frac{1}{\beta}-1}}{\alpha^{\frac{1}{\beta}}} \Gamma\left(\frac{1}{\beta}, \frac{\alpha}{\beta}t^\beta\right), \quad (10)$$

where $\Gamma(a,\ x)$ is incomplete gamma function:

$$\Gamma(a,\ x) = \frac{1}{\Gamma(a)} \int_x^{+\infty} e^{-t} t^{a-1} dt.$$

The presentation (2) and formula (10) give

$$\sigma^2(t) = \frac{2}{e^{\frac{\alpha}{\beta}} t^{\beta-1} e^{-\frac{\alpha}{\beta}t^\beta}} \frac{e^{\frac{\alpha}{\beta}}}{\alpha} \frac{\beta^{\frac{1}{\beta}-1}}{\alpha^{\frac{1}{\beta}}} \Gamma\left(\frac{1}{\beta}, \frac{\alpha}{\beta}t^\beta\right) - \frac{t^{2-2\beta}}{\alpha^2}.$$

Thus, for the residual variance, an exact equality is obtained with using an incomplete gamma function:

$$\sigma^2(t) = \frac{2t^{1-\beta}}{e^{-\frac{\alpha}{\beta}t^\beta}} \frac{\beta^{\frac{1}{\beta}-1}}{\alpha^{\frac{1}{\beta}+1}} \Gamma\left(\frac{1}{\beta}, \frac{\alpha}{\beta}t^\beta\right) - \frac{t^{2-2\beta}}{\alpha^2}. \quad (11)$$

Formula 8.357 from [5] leads immediately to the expansion of incomplete gamma function in (10)

$$\Gamma\left(\frac{1}{\beta}, \frac{\alpha}{\beta}t^\beta\right) =$$

$$= \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta}-1} t^{1-\beta} e^{-\frac{\alpha}{\beta}t^\beta} \left(1 - \frac{\beta\left(1 - \frac{1}{\beta}\right)}{\alpha} t^{-\beta} + \frac{\beta^2\left(1 - \frac{1}{\beta}\right)\left(2 - \frac{1}{\beta}\right)}{\alpha^2} t^{-2\beta} + O\left(t^{-3\beta}\right)\right),$$

as $t \to \infty$.

By inserting asymptotics from the above in (11) it follows that

$$\sigma^2(t) =$$

$$= 2\frac{t^{2-2\beta}}{\alpha}\left(1 - \frac{(\beta-1)}{\alpha}t^{-\beta} + \frac{(\beta-1)(2\beta-1)}{\alpha^2}t^{-2\beta} + O\left(t^{-3\beta}\right)\right) - \frac{t^{2-2\beta}}{\alpha^2} =$$

$$= \frac{t^{2-2\beta}}{\alpha^2}\left(1 - 2\frac{(\beta-1)}{\alpha}t^{-\beta} + 2\frac{(\beta-1)(2\beta-1)}{\alpha^2}t^{-2\beta} + O\left(t^{-3\beta}\right)\right), \quad \text{as } t \to \infty.$$

Finally it yields the following asymptotic result for the residual variance of Benktander type II distribution

$$\sigma^2(t) = \frac{t^{2-2\beta}}{\alpha^2}\left(1 - 2\frac{(\beta-1)}{\alpha}t^{-\beta} + 2\frac{(\beta-1)(2\beta-1)}{\alpha^2}t^{-2\beta} + O\left(t^{-3\beta}\right)\right), \quad \text{as } t \to \infty, \quad (12)$$

and residual standard deviation

$$\sigma(t) = \frac{t^{1-\beta}}{\alpha}\left(1 - \frac{(\beta-1)}{\alpha}\frac{1}{t^{\beta}} + \frac{(\beta-1)(2\beta-1)}{\alpha^2}\frac{1}{t^{2\beta}} + O\left(\frac{1}{t^{3\beta}}\right)\right), \quad \text{as } t \to \infty. \quad (13)$$

2.4. Burr Distribution

The Burr distribution or Burr Type XII distribution (see [6]) is popular as a fit model of a set insurance data. The following representations are obtained for the Burr distribution moments:

$$\mu(t) = \frac{t}{ck-1}\left[1 + \frac{ck}{ck-1+c}\cdot\frac{1}{t^c} + \frac{ck(1-c)}{(ck-1+c)(ck-1+2c)}\cdot\frac{1}{t^{2c}} + o\left(\frac{1}{t^{2c}}\right)\right];$$

$$\sigma^2(t) = \frac{ck}{(ck-1)^2}\cdot t^2\cdot\left[\frac{1}{ck-2} + \frac{2c(k+1)}{(ck-1+c)(ck-2+c)}\cdot\frac{1}{t^c} + o\left(\frac{1}{t^c}\right)\right], \quad \text{as } t \to \infty.$$

3. Conclusions

We illustrate the behavior of the empirical composite samples for some simulated data sets of heavy-tailed distributions in Section 1. Explicit accurate innovated asymptotic expansions are found of the MRL (ME) and residual variance for well known models from the (Table 3.4.7 , [1], p. 161). By application of criterion of Balkema; de Haan [7] one may found the belongs (or not) of these distributions to domain of attraction exponential distribution.

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Abbreviations

The following abbreviations are used in this manuscript:

DF	Cumulative distribution function
MRL	Mean residual life
ME	Mean excess function
PDF	Probability density function

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