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Gauss-Seidel and Sor Methods for Solving Intuitionistic Fuzzy System of Linear Equations †

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Abstract: Solving various real-life problems ultimately requires solving systems of linear equations. However, the parameters involved in such real-life problems may be pervaded with uncertainty, which result in fuzzy parameters rather than crisp parameters. Intuitionistic fuzzy parameters are more suitable for some cases since they allow to deal with the feeling of fear or hesitation when making a decision, which are characteristics of human being in applying knowledge and skills. Intuitionistic fuzzy linear system (IFLS) resulting from the real-life problem involves large number of equations and equally large number of unknowns. When IFLS is in matrix-vector form the resulting coefficient matrix will have a sparse structure, which makes iterative methods necessary for their solution. In this paper, the known Gauss-Seidel and SOR iterative methods for solving linear system of equations are discussed, to the best of our knowledge for the first time, to solve (IFLS). The single parametric form representation of intuitionistic fuzzy numbers (IFN) makes it possible to apply these iterative techniques to IFLS. Finally a problem of voltage input output in electric circuit has been considered to show the applicability and the efficiency of these methods.

Keywords: Parametric form of Intuitionistic Fuzzy Number; Intuitionistic fuzzy linear system (IFLS); Gauss-Seidel and SOR Iterative method

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1. Introduction

In the real world, many of our scientific problems turn into problems of solving linear system of equations. Parameters involved in such equations are generally determined through some estimation, experiment and modeling. So the parameters often involve some uncertainty or impreciseness in them. Therefore, our preferred choice is to choose fuzzy parameters rather than crisp parameters. Intuitionistic fuzzy parameters are more flexible in describing uncertainty with membership and non-membership functions with hesitancy function than fuzzy parameters. To handle this uncertainty or impreciseness, Zadeh [1] introduced the concept of fuzzy set theory. Since then, there have been several generalizations of fuzzy set theory made by researchers. One of them is intuitionistic fuzzy set theory, which was introduced by Atanassov [2,3]. Friedman et al. [4] proposed a general model to solve $n \times n$ FSLE in which the coefficient matrix is crisp and the right-hand side is an arbitrary fuzzy vector. Iterative methods for solving FSLE are given by Allahviranloo [5]. The SOR method to solve FSLE given by Allahviranloo [6]. To solve IFLS, several authors provided different approaches. Atti et al. [7] developed an approach to solve IFLS, in which they converted $n \times n$ IFLS into four $n \times n$ crisp linear systems of equations. Saw et al. [8] proposed the Jacobi iterative method to solve IFLS. They converted $n \times n$ IFLS into one $4n \times 4n$ crisp linear system of equations. In the present work, we extended the well-known Gauss-Seidel and SOR methods to solve IFLS.

2. Materials and Methods

The $n \times n$ intuitionistic fuzzy system of linear equations may be written as

$$\begin{array}{c} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \ldots + a_{1n}\tilde{x}_n = \tilde{y}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots + a_{2n}\tilde{x}_n = \tilde{y}_2, \\ \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \ldots + a_{nn}\tilde{x}_n = \tilde{y}_n. \end{array} \quad (1)$$

In matrix-vector form the above system may be written as $A\tilde{X} = \tilde{Y}$, where the coefficient matrix $A = (a_{ij}),\ 1 \leq i \leq n,\ 1 \leq j \leq n$ is a crisp real $n \times n$ matrix, $\tilde{Y} = (\tilde{y}_i),\ 1 \leq i \leq n$, is a column vector of fuzzy numbers and $\tilde{X} = (\tilde{x}_j),\ 1 \leq j \leq n$, is the vector of fuzzy unknowns.

Definition 1. *An intuitionistic fuzzy number vector $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)^t$ given by $(\tilde{x}_j = (\underline{x_j}^+(\alpha), \overline{x_j}^+(\alpha)), (\underline{x_j}^-(\alpha), \overline{x_j}^-(\alpha))), 1 \leq j \leq n, 0 \leq \alpha \leq 1$, is called solution of (1) if:*

$$\underline{\sum_{j=1}^n a_{ij} x_j{}^+} = \sum_{j=1}^n \underline{a_{ij} x_j{}^+} = \underline{y_i}^+, i = 1,2,...,n, \overline{\sum_{j=1}^n a_{ij} x_j{}^+} = \sum_{j=1}^n \overline{a_{ij} x_j{}^+} = \overline{y_i}^+, i = 1,2,...,n,$$
$$\underline{\sum_{j=1}^n a_{ij} x_j{}^-} = \sum_{j=1}^n \underline{a_{ij} x_j^-} = \underline{y_i}^-, i = 1,2,...,n, \overline{\sum_{j=1}^n a_{ij} x_j{}^-} = \sum_{j=1}^n \overline{a_{ij} x_j^-} = \overline{y_i}^-, i = 1,2,...,n.$$

Hence from (1) we have four crisp $n \times n$ linear systems for all i which can be extended to a $4n \times 4n$ crisp linear system as follows: $SX = Y \Longrightarrow \begin{pmatrix} S_1 & S_2 & 0 & 0 \\ S_2 & S_1 & 0 & 0 \\ 0 & 0 & S_1 & S_2 \\ 0 & 0 & S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix} = \begin{pmatrix} \underline{Y}_\alpha \\ \overline{Y}_\alpha \\ \underline{Y}^\alpha \\ \overline{Y}^\alpha \end{pmatrix}$,

where s_{ij} are determined as follows:

$$\begin{array}{l} a_{ij} \geq 0 \Rightarrow s_{ij} = s_{i+n,j+n} = s_{i+2n,j+2n} = s_{i+3n,j+3n} = a_{ij}, \\ a_{ij} \leq 0 \Rightarrow s_{i,j+n} = S_{i+n,j} = s_{i+2n,j+3n} = s_{i+3n,j+2n} = a_{ij}, \end{array}$$

and s_{ij} which are not determined are zero.

Also, $\underline{X}_\alpha = \begin{pmatrix} \underline{x_1}^+ \\ \underline{x_2}^+ \\ \vdots \\ \underline{x_n}^+ \end{pmatrix}, \overline{X}_\alpha = \begin{pmatrix} \overline{x_1}^+ \\ \overline{x_2}^+ \\ \vdots \\ \overline{x_n}^+ \end{pmatrix}, \underline{X}^\alpha = \begin{pmatrix} \underline{x_1}^- \\ \underline{x_2}^- \\ \vdots \\ \underline{x_n}^- \end{pmatrix}, \overline{X}^\alpha = \begin{pmatrix} \overline{x_1}^- \\ \overline{x_2}^- \\ \vdots \\ \overline{x_n}^- \end{pmatrix}$ and

$\underline{Y}_\alpha = \begin{pmatrix} \underline{y_1}^+ \\ \underline{y_2}^+ \\ \vdots \\ \underline{y_n}^+ \end{pmatrix}, \overline{Y}_\alpha = \begin{pmatrix} \overline{y_1}^+ \\ \overline{y_2}^+ \\ \vdots \\ \overline{y_n}^+ \end{pmatrix}, \underline{Y}^\alpha = \begin{pmatrix} \underline{y_1}^- \\ \underline{y_2}^- \\ \vdots \\ \underline{y_n}^- \end{pmatrix}, \overline{Y}^\alpha = \begin{pmatrix} \overline{y_1}^- \\ \overline{y_2}^- \\ \vdots \\ \overline{y_n}^- \end{pmatrix}$.

From the structure of S, it is clear that S_1 contains the positive entries of the matrix A, S_2 contains the negative entries of the matrix A and $A = S_1 + S_2$. We represent S as $S = \begin{pmatrix} S_D & \bar{0} \\ \bar{0} & S_D \end{pmatrix}$, where $S_D = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ and $\bar{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Theorem 1. *Let the matrix S be strictly diagonally dominant then Gauss-Seidel iterate converges to $S^{-1}Y$ for any X^0. (see [9], p. 120)*

Theorem 2. *The matrix S is non-singular iff $A = S_1 + S_2$ and $(S_1 - S_2)$ are both non-singular. (see [4])*

Proof. The matrix S is non singular iff S_D is non singular. Now $S_D = \begin{pmatrix} S_1 \geq 0 & S_2 \leq 0 \\ S_2 \leq 0 & S_1 \geq 0 \end{pmatrix}$ is non singular iff $A = S_1 + S_2$ and $(S_1 - S_2)$ is non singular. □

Theorem 3. *Let S be non singular. Then the unique solution X of Equation (1) is always a intuitionistic fuzzy vector for arbitrary vector Y, if S^{-1} is nonnegative. (see [5])*

Theorem 4. *The matrix A in Equation (1) is strictly diagonally dominant iff the matrix S be strictly diagonally dominant. (see [8])*

2.1. Gauss-Seidel Iterative Scheme:

Without loss of generality suppose that $s_{ii} > 0$ for all $i = 1, 2, ..., 4n$.
Let $S = D + L + U$ where

$$D = \begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & D_1 \end{pmatrix}, L = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ S_2 & L_1 & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & S_2 & L_1 \end{pmatrix}, U = \begin{pmatrix} U_1 & S_2 & 0 & 0 \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & U_1 & S_2 \\ 0 & 0 & 0 & U_1 \end{pmatrix}$$

$(D_1)_{ii} = s_{ii} > 0, i = 1, 2, ..., n$ and suppose $S_1 = D_1 + L_1 + U_1$.

From $SX = Y$ we have

$$\begin{pmatrix} D_1+L_1 & 0 & 0 & 0 \\ S_2 & D_1+L_1 & 0 & 0 \\ 0 & 0 & D_1+L_1 & 0 \\ 0 & 0 & S_2 & D_1+L_1 \end{pmatrix} \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix} + \begin{pmatrix} U_1 & S_2 & 0 & 0 \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & U_1 & S_2 \\ 0 & 0 & 0 & U_1 \end{pmatrix} \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix} = \begin{pmatrix} \underline{Y}_\alpha \\ \overline{Y}_\alpha \\ \underline{Y}^\alpha \\ \overline{Y}^\alpha \end{pmatrix}$$

Then,

$$\begin{aligned} \underline{X}_\alpha &= (D_1 + L_1)^{-1}\underline{Y}_\alpha - (D_1 + L_1)^{-1}U_1\underline{X}_\alpha - (D_1 + L_1)^{-1}S_2\overline{X}_\alpha \\ \overline{X}_\alpha &= (D_1 + L_1)^{-1}\overline{Y}_\alpha - (D_1 + L_1)^{-1}U_1\overline{X}_\alpha - (D_1 + L_1)^{-1}S_2\underline{X}_\alpha \\ \underline{X}^\alpha &= (D_1 + L_1)^{-1}\underline{Y}^\alpha - (D_1 + L_1)^{-1}U_1\underline{X}^\alpha - (D_1 + L_1)^{-1}S_2\overline{X}^\alpha \\ \overline{X}^\alpha &= (D_1 + L_1)^{-1}\overline{Y}^\alpha - (D_1 + L_1)^{-1}U_1\overline{X}^\alpha - (D_1 + L_1)^{-1}S_2\underline{X}^\alpha \end{aligned}$$

So, the Gauss-Seidel iterative technique reads as:

$$\begin{aligned} (\underline{X}_\alpha)^{k+1} &= (D_1 + L_1)^{-1}\underline{Y}_\alpha - (D_1 + L_1)^{-1}U_1(\underline{X}_\alpha)^k - (D_1 + L_1)^{-1}S_2(\overline{X}_\alpha)^k \\ (\overline{X}_\alpha)^{k+1} &= (D_1 + L_1)^{-1}\overline{Y}_\alpha - (D_1 + L_1)^{-1}U_1(\overline{X}_\alpha)^k - (D_1 + L_1)^{-1}S_2(\underline{X}_\alpha)^k \\ (\underline{X}^\alpha)^{k+1} &= (D_1 + L_1)^{-1}\underline{Y}^\alpha - (D_1 + L_1)^{-1}U_1(\underline{X}^\alpha)^k - (D_1 + L_1)^{-1}S_2(\overline{X^\alpha})^k \\ (\overline{X}^\alpha)^{k+1} &= (D_1 + L_1)^{-1}\overline{Y}^\alpha - (D_1 + L_1)^{-1}U_1(\overline{X}^\alpha)^k - (D_1 + L_1)^{-1}S_2(\underline{X}^\alpha)^k \end{aligned}$$

The result in the matrix-vector form of the Gauss-Seidel iterative technique are $X^{(k+1)} = M_{GS}X^k + C$ where

$$M_{GS} = \begin{pmatrix} -(D_1 + L_1)^{-1}U_1 & -(D_1 + L_1)^{-1}S_2 & 0 & 0 \\ -(D_1 + L_1)^{-1}S_2 & -(D_1 + L_1)^{-1}U_1 & 0 & 0 \\ 0 & 0 & -(D_1 + L_1)^{-1}U_1 & -(D_1 + L_1)^{-1}S_2 \\ 0 & 0 & -(D_1 + L_1)^{-1}S_2 & -(D_1 + L_1)^{-1}U_1 \end{pmatrix},$$

$$C = \begin{pmatrix} (D_1 + L^1)^{-1}\underline{Y}_\alpha \\ (D_1 + L^1)^{-1}\overline{Y}_\alpha \\ (D_1 + L^1)^{-1}\underline{Y}^\alpha \\ (D_1 + L^1)^{-1}\overline{Y}^\alpha \end{pmatrix}, X = \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix}.$$

From Theorem (1) and (4), the Gauss-Seidel iterates converge to the unique solution $X = S^{-1}Y$, for any X^0. The stopping criterion with tolerance $\epsilon > 0$ is
$\frac{\|(\underline{X}_\alpha)^{k+1} - (\underline{X}_\alpha)^k\|}{\|(\underline{X}_\alpha)^{k+1}\|} < \epsilon$, $\frac{\|(\overline{X}_\alpha)^{k+1} - (\overline{X}_\alpha)^k\|}{\|(\overline{X}_\alpha)^{k+1}\|} < \epsilon$, $\frac{\|(\underline{X}^\alpha)^{k+1} - (\underline{X}^\alpha)^k\|}{\|(\underline{X}^\alpha)^{k+1}\|} < \epsilon$, $\frac{\|(\overline{X}^\alpha)^{k+1} - (\overline{X}^\alpha)^k\|}{\|(\overline{X}^\alpha)^{k+1}\|} < \epsilon$.

2.2. SOR Iterative Scheme:

If we decomposed S_1 matrix as $S_1 = D_1 + L_1 + U_1$, with diagonal component D_1, and strictly lower triangular component L_1 and upper triangular component U_1, then decomposed matrix S becomes as $S = D + L + U$ where

$$D = \begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & D_1 \end{pmatrix}, L = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ S_2 & L_1 & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & S_2 & L_1 \end{pmatrix}, U = \begin{pmatrix} U_1 & S_2 & 0 & 0 \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & U_1 & S_2 \\ 0 & 0 & 0 & U_1 \end{pmatrix}$$

From $SX = Y$ we rewrite the system as,

$$(D + L + U)X = Y \tag{2}$$

Using relaxation parameter ω, we rewrite the above system in new form as,

$$(D + \omega L)X = \omega Y - [(\omega - 1)D + \omega U]X, \tag{3}$$

$$\begin{pmatrix} D_1 + \omega L_1 & 0 & 0 & 0 \\ \omega S_2 & D_1 + \omega L_1 & 0 & 0 \\ 0 & 0 & D_1 + \omega L_1 & 0 \\ 0 & 0 & \omega S_2 & D_1 + \omega L_1 \end{pmatrix} \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix} =$$

$$\omega \begin{pmatrix} \underline{Y}_\alpha \\ \overline{Y}_\alpha \\ \underline{Y}^\alpha \\ \overline{Y}^\alpha \end{pmatrix} - \begin{pmatrix} (\omega-1)D_1 + \omega U_1 & \omega S_2 & 0 & 0 \\ 0 & (\omega-1)D_1 + \omega U_1 & 0 & 0 \\ 0 & 0 & (\omega-1)D_1 + \omega U_1 & \omega S_2 \\ 0 & 0 & 0 & (\omega-1)D_1 + \omega U_1 \end{pmatrix} \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix}$$

Then we get,

$$\begin{aligned} \underline{X}_\alpha &= (D_1 + \omega L_1)^{-1}\omega \underline{Y}_\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1]\underline{X}_\alpha - (D_1 + \omega L_1)^{-1}S_2\overline{X}_\alpha \\ \overline{X}_\alpha &= (D_1 + \omega L_1)^{-1}\omega \overline{Y}_\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1]\overline{X}_\alpha - (D_1 + \omega L_1)^{-1}S_2\underline{X}_\alpha \\ \underline{X}^\alpha &= (D_1 + \omega L_1)^{-1}\omega \underline{Y}^\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1]\underline{X}^\alpha - (D_1 + \omega L_1)^{-1}S_2\overline{X}^\alpha \\ \overline{X}^\alpha &= (D_1 + \omega L_1)^{-1}\omega \overline{Y}^\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1]\overline{X}^\alpha - (D_1 + \omega L_1)^{-1}S_2\underline{X}^\alpha \end{aligned}$$

So, the SOR iterative technique read as:

$$\begin{aligned} (\underline{X}_\alpha)^{k+1} &= (D_1 + \omega L_1)^{-1}\omega \underline{Y}_\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1](\underline{X}_\alpha)^k - (D_1 + \omega L_1)^{-1}S_2(\overline{X}_\alpha)^k \\ (\overline{X}_\alpha)^{k+1} &= (D_1 + \omega L_1)^{-1}\omega \overline{Y}_\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1](\overline{X}_\alpha)^k - (D_1 + \omega L_1)^{-1}S_2(\underline{X}_\alpha)^k \\ (\underline{X}^\alpha)^{k+1} &= (D_1 + \omega L_1)^{-1}\omega \underline{Y}^\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1](\underline{X}^\alpha)^k - (D_1 + \omega L_1)^{-1}S_2(\overline{X}^\alpha)^k \\ (\overline{X}^\alpha)^{k+1} &= (D_1 + \omega L_1)^{-1}\omega \overline{Y}^\alpha - (D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega - 1)D_1](\overline{X}^\alpha)^k - (D_1 + \omega L_1)^{-1}S_2(\underline{X}^\alpha)^k \end{aligned}$$

This can be written in matrix-vector form as, $X^{(k+1)} = M_{SOR}X^k + C$ where

$$M_{SOR} = \begin{pmatrix} -(D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega-1)D_1] & -(D_1 + \omega L_1)^{-1}S_2 & 0 & 0 \\ -(D_1 + \omega L_1)^{-1}S_2 & -(D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega-1)D_1] & 0 & 0 \\ 0 & 0 & -(D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega-1)D_1] & -(D_1 + \omega L_1)^{-1}S_2 \\ 0 & 0 & -(D_1 + \omega L_1)^{-1}S_2 & -(D_1 + \omega L_1)^{-1}[\omega U_1 + (\omega-1)D_1] \end{pmatrix},$$

$$C = \begin{pmatrix} (D_1 + L_1)^{-1}\underline{Y}_\alpha \\ (D_1 + L_1)^{-1}\overline{Y}_\alpha \\ (D_1 + L_1)^{-1}\underline{Y}^\alpha \\ (D_1 + L_1)^{-1}\overline{Y}^\alpha \end{pmatrix}, X = \begin{pmatrix} \underline{X}_\alpha \\ \overline{X}_\alpha \\ \underline{X}^\alpha \\ \overline{X}^\alpha \end{pmatrix}.$$

3. A Practical Application

The authors in [10] considered the electrical circuit shown in Figure 1, where $\tilde{v_1}$ and $\tilde{v_2}$ are the input voltages, and $\tilde{v_3}$ and $\tilde{v_4}$ are the output voltages. The circuit is a kind of summing amplifier with two inputs and two outputs. The relationship between input and output voltages is as follows: $\begin{pmatrix} 3 & 0.5 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} \tilde{v_1} \\ \tilde{v_2} \end{pmatrix} = \begin{pmatrix} \tilde{v_3} \\ \tilde{v_4} \end{pmatrix}$.

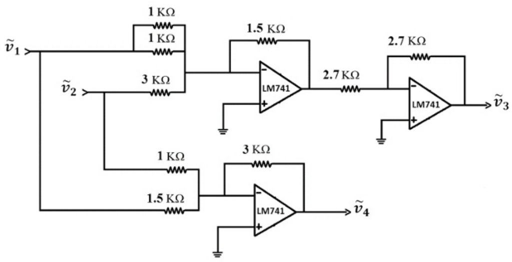

Figure 1. Electrical Circuit.

They considered the output voltages as type-2 fuzzy numbers. In this paper we treat the same example but we consider the output voltages as intuitionistic fuzzy numbers as considered by the authors in [7]:
$\tilde{v_3} = (14 + 2\alpha; -14 - 2\alpha; 16 - 3\alpha; 16 + 3\alpha)$ and
$\tilde{v_4} = (-18 + 2\alpha; -14 - 2\alpha; -16 - 3\alpha; -16 + 3\alpha)$.

Here we are looking at how to calculate the input voltages when the output voltages are known but uncertain. That is, $\tilde{v_3}$ is "about 16 volts" and $\tilde{v_4}$ is "about –16 volts". Different experts may have different viewpoints on the output voltage's uncertainty.

Now, if we choose to focus on one expert's interpretation, then the linear system shown in Equation (1) will be type-1 FLSE, as mentioned in [10].

In addition, if we consider the hesitation of the expert, which is quite natural when making a decision, because of characteristics of human beings in applying knowledge and skills. Then the linear system shown in Equation (1) will be an intuitionistic fuzzy linear system of equations. This is more realistic than type-1 FLSE.

If we want to consider more than one expert's opinion, then we get the system of equations as type-2 FLSE, originally considered in [10].

Now, if we consider different expert's opinion individually together with their hesitations, then we can take, for example, the arithmetic average of different IFNs to get output voltages, and the system can be better represented as IFLS.

In this case the above system reduces to

$$\begin{cases} 3\tilde{v_1} + 0.5\tilde{v_2} = (14 + 2\alpha; 18 - 2\alpha; 16 - 3\alpha; 16 + 3\alpha) \\ -2\tilde{v_1} - 3\tilde{v_2} = (-18 + 2\alpha; -14 - 2\alpha; -16 - 3\alpha; -16 + 3\alpha) \end{cases} \quad (4)$$

The exact and approximated solutions are plotted and compared for $\tilde{v_1}$.

The exact and approximated solutions are plotted and compared for $\tilde{v_2}$.

4. Conclusions

As can be seen in Figures 2 and 3, the solutions obtained by both the method for tolerance $\epsilon = 10^{-6}$ agreed quite well with the exact solution for both $\tilde{v_1}$ and $\tilde{v_2}$. The convergence history in Figure 4 shows that Gauss-Seidel method requires 9 iterations and SOR method requires 7 iterations to converge in this case. As expected, SOR method with $w_{opt} = 0.9$ is faster than the Gauss-Seidel method, even for this relatively small $(for\, n = 2\ IFLS,$ i.e., $8 \times 8\ crisp)$ system of equations. Certainly, for large system of equations convergence will be accelerated using SOR method than Gauss-Seidel method.

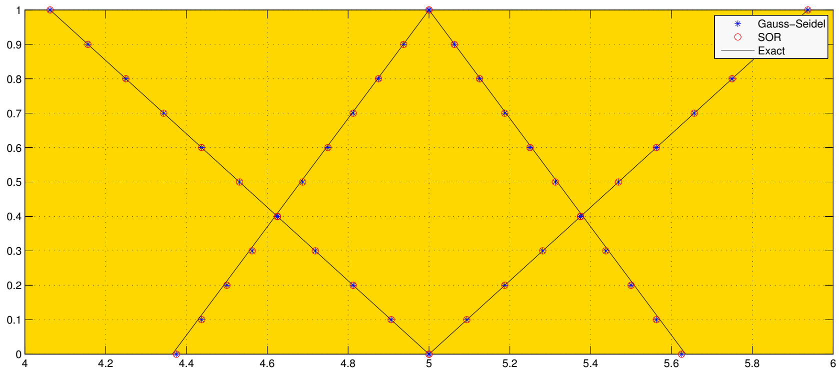

Figure 2. Graphical representation of $\tilde{v}_1$ with continuous (exact solution) and approximate values for $\alpha = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ for both Gauss-Seidel and SOR methods.

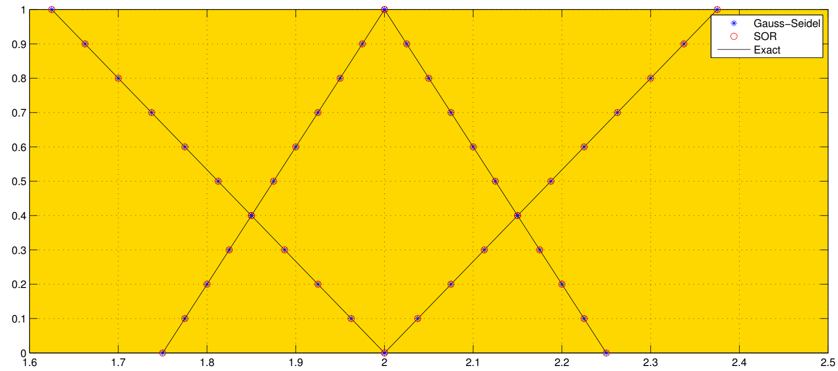

Figure 3. Graphical representation of $\tilde{v}_2$ with continuous (exact solution) and approximate values for $\alpha = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ for both Gauss-Seidel and SOR methods.

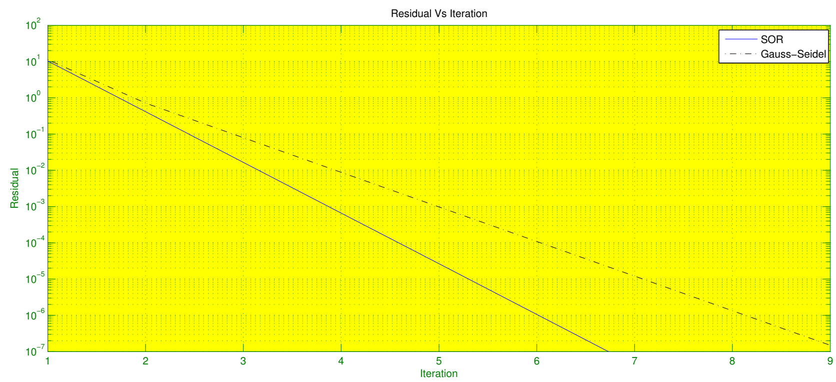

Figure 4. Convergence history of Gauss-Seidel and SOR methods.

In future, we try to accelerate the convergence of the linear system using more efficient iterative methods such as Krylov subspace methods or Multigrid methods.

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