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On the Stability and Controllability of a Degenerate Differential Systems in Banach Spaces †

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Abstract: The aim of this research is to generalize the famous Lyapunov theorem of the classical explicit differential systems (Continuous or Discrete) given by the two abstract forms: $x'(t) = Px(t)$, $x_{n+1} = Px_n$, where P is a linear operator or a matrix if the space has finite dimension, in order to study the spectrum of a degenerate differential systems as $Ax'(t) = Bx(t), t \geq 0$, where A and B are two linear bounded operators in Hilbert spaces and A is not an invertible operator. Using some properties of the spectral theory for the pencil $\lambda A - B$ which is obtained by substituting $x(t) = e^{\lambda t}v$ in the homogenous differential equation of the previuos degenerate system, and an appropriate conformal mapping. The achieved results can be applied to study the stability and controllability of certain implicit differential systems.

Keywords: pencil of operators; stabilizability; controllability

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1. Introduction

The fondamental challenges of mathematical problems about differential systems appears precisely in cotrol theory, many researchers use linear systems of two types: continuous or discrete. Since 1970, many mathematiciens become intersted in another wide classes of differential systems which have a real pratical and physical applications can be found in [1–4].

In the present paper, we propose the problem differential of implicit stationary equation discribed as follows:

$$Ax'(t) = Bx(t) + \varphi(t, x(t)), \quad t \geq 0; \tag{1}$$

with initial condition

$$x(t_0) = x_0.$$

Here, we assume that A and B are two bounded linear operators acting in the same complex Hilbert space $\mathcal{H}$ and φ is a continuous function from $[0, \infty[\times \mathcal{H}$ into $\mathcal{H}$, the operator A is not invertible.

We need those notations: $\|.\|$, $< ., . >$ are the norm and the inner product in $\mathcal{H}$. Now, we consider the system (1) in the homogenous case and suppose that it has a solution then:

- The system (1) is said to be exponential, if for any solution $x(t)$ with $t \geq 0$, we have

$$\|x(t)\| \leq Me^{\alpha t}\|x_0\|, \quad t \geq 0; \tag{2}$$

  where the constants α and M are not depanded on the solution $x(t)$.
- For $\alpha < 0$, the system (1) is exponentially stable. In particular, for $\alpha = 0$ it is uniformly bounded.

- The system (1) is well-posed, if it satisfies the following conditions:
  1. If for any solution $x(.)$ such that $x(0) = x_0$, then $x(t) = 0$ for all $t \geq 0$;
  2. it generates an evolution semigroup of bounded operators $S(t) : x_0 \longmapsto x(t)$ for all $t \geq 0$.

  The operators $S(t)$ are defined on the set $D_0 = \{x_0\}$ of the admissible initial vectors x_0.

2. Lyapunov's Theorem and Its Generalization

In this section, we investigate the homogeneous problem of Equation (1) with $\varphi \equiv 0$ in the next form:

$$Ax'(t) = Bx(t), \quad t \geq 0, \ x(t_0) = x_0; \tag{3}$$

our main obective is to extand the general Lyapunov theorem [1] of the linear differential systems to the operator pencil $\lambda A - B$ then, we apply the achieved results to affirme the stability and controllability of some degenerate systems as Equation (3).

Definition 1 ([3])**.** *The complex parameter λ is said to be regular point of the pencil $\lambda A - B$, if the operator $(\lambda A - B)^{-1}$ exists and it is bounded in $\mathcal{H}$.*

We denote by $\rho(A, B)$ the set of all regular points and its complement by $\sigma(A, B) = \mathbb{C} \backslash \sigma_p(A, B)$, which is also called the spectrum of the operator pencil $\lambda A - B$. The set of all eigen-values of the pencil $\lambda A - B$ is denoted by σ_p, such that

$$\sigma_p(A, B) = \{\lambda \in \mathbb{C} | \exists v \neq 0, (\lambda A - B)v = 0\}.$$

Theorem 1. *Consider the problem (3). If the spectrum $\sigma(A, B)$ of the linear bounded operators A and B lies in the left half plane ($Rel(\lambda) < 0$), then for any uniform positive operator $G \gg 0$, there exists a uniform positive operator $W \gg 0$ such that*

$$B^*WA + A^*WB = G. \tag{4}$$

Proof. We suppose that $\sigma(A, B) \subset \{\lambda : Re(\lambda) < 0\}$, then i is a regular point and the operator $T = i(iA + B)(iA - B)^{-1}$ is bounded. Now, we use the conformal mapping $\mu = \phi(\lambda) = \dfrac{-i\lambda + 1}{\lambda - i}$, we obtain

$$T - \mu I = \frac{-2}{z - i}(\lambda A - B)(iA - B)^{-1}$$

So, the operator $T - \mu I$ is invertible if and only if the pencil $\lambda A - B$ is also invertible. Hence,

$$\sigma(T) = \sigma(I, T) = \phi(\sigma(A, B)).$$

Using the classical Lyapunov's theorem [3], we have:
For any operator $H \gg 0$, there exists an operator $W \gg 0$ such that

$$\begin{aligned} Re(WT) &= \frac{WT + T^*W}{2} \\ &= \frac{1}{2}(iW(iA + B)(iA - B)^{-1} - i(-iA^* - B^*)^{-1}(-iA^* + B^*)W) \\ &= (-iA^* - B^*)^{-1}(A^*WB + B^*WA)(iA - B)^{-1} \\ &= -H. \end{aligned}$$

We put $G = B^*WA + A^*WB$, with $G = -(iA^* + B^*)H(iA - B) \gg 0$.
In fact, $G = G^*$ and $< Gx, x > \geq c\|x\|^2, c > 0$. Thus, our theorem is proved. □

Theorem 2. *If Equation (4) is satisfying for the pair(W, G) of bounded positive unifor operators, then i is not an eigen-value for the operator pencil $\lambda A - B$.*

Proof. Suppose that $i \in \sigma_p(A, B)$, and v is its eigen-vector then, $(iA - B)v = 0$ also $Bv = iAv$. After the computing of the scalar product, we obtain

$$< Gv, v >=< (B^*WA + A^*WB)v, v >= 0, \quad \forall v \in \mathcal{H}. \tag{5}$$

Since $G \gg 0$, then $< Gv, v >\geq c\|v\|^2 > 0$ which shows a contraduction with the main hypothesis above. □

We recall some notes conserning uncontrollable system as (3) before summarizing our results. Let $D_0 = \{x_0\}$ be the initial subspace of $\mathcal{H}$, we denote by $A_0 = A\backslash D_0$ the invertible restriction of the operator A in D_0 (The operator A_0 is invertible, if the system (3) is well-posed).

Lemma 1. *Let A_0 be an invertible operator. If $\varphi(\tau, x(\tau)) \in AD_0$ for any $\tau \geq \tau_0$ and the function $S(t-\tau)A_0^{-1}\varphi(\tau, x(\tau))$ is integrable (with respect to τ) where $\{S(t)\}_{t\geq 0}$ is the semigroup of the operators of Equation (3), then the system (3) is equivalent to the following equation:*

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)A_0^{-1}\varphi(\tau, x(\tau))d\tau. \tag{6}$$

Lemma 2 (See ([3])). *If $g(t) \leq c + \int_0^t g(\tau)h(\tau)$ for all $t \geq 0$, where h is a continuous positive real function and $c > 0$ is an arbitrary constant, then*

$$g(t) \leqslant c.exp(\int_0^t h(\tau)d\tau).$$

Theorem 3. *Suppose that:*

1. *the system (3) is well-posed;*
2. *the linear operator $\varphi(t, x(t))$ tronsforms D_0 into AD_0 such that*

$$\int_0^\infty \|A_0^{-1}\varphi(t, x(t))\|dt < \infty, \quad \forall t \geq 0;$$

then the system (3) is exponentiall stable.

Proof. Assuming that the first condition of this theorem is verified, then we have

$$\|S(t)x_0\| \leq Me^{\alpha t}\|x_0\|,$$

and

$$\|S(t-\tau)A_0^{-1}\varphi(\tau, x(\tau))\| \leq Me^{t-\tau}\|A_0^{-1}\varphi(\tau, x(\tau))\|,$$

with $A_0^{-1}\varphi(\tau, x(\tau)) \in D_0$. Using Equation (6), then we obtain

$$\|x(t)\| \leq Me^{\alpha t}\|x_0\| + M\int_0^t e^{\alpha(t-\tau)}\|A_0^{-1}\varphi(\tau, x(\tau))\|\|x(\tau)\|d\tau,$$

which is equivalent to

$$e^{-\alpha t}\|x(t)\| \leq M\|x_0\| + M\int_0^t e^{-\alpha\tau}\|A_0^{-1}\varphi(\tau, x(\tau))\|\|x(\tau)\|d\tau.$$

Applying Lemma 2 where $g(t) = e^{-\alpha t}\|x(t)\|$, and $h(\tau) = M\|A_0^{-1}\varphi(\tau, x(\tau))$, $c = M\|x_0\|$, then

$$e^{-\alpha t\|x(t)}\| \leq M\|x_0\|exp[M\int_0^t \|A_0^{-1}\varphi(\tau, x(\tau))\|d\tau] \leq M\|x_0\|exp[M\int_0^\infty \|A_0^{-1}\varphi(\tau, x(\tau))\|d\tau].$$

Therefore, $\|x(t)\| \leq M_1 e^{\alpha t} x_0$. □

Particulary, in finite dimentional spaces we can use the theory of elementary divisors of the matrix pencil $\lambda A - B$ for example see [5], we establish the next important result.

Theorem 4. *The following statement are equivalent:*

1. *The system* (3) *is exponentially stable;*
2. $\sigma(A, B) = \sigma(A, B) \subset \{\lambda : Re(\lambda) < 0\}$;
3. *There exists a positive definite matrix* $W \gg 0$ *such that* $B^*WA + A^*WB \gg 0$.

Remark 1. *In finite dimentional spaces, exponential stability is characterized by the fact that the spectrum of matrices A and B lies in the left half plane, but the situation in infinite dimentional spaces is much more complicated.*

3. Relation between Stabilizability and Controllability

We provide here some definition and basic reluts about the exact controllability and complete stabilizability of an implicit differential control system governed by the general form:

$$Ax'(t) = Bx(t) + Cu(t), \quad x(0) = x_0. \tag{7}$$

where C is also a linear bounded operator and $u(t)$ is a function takes values in the Hilbert space $U \subset \mathcal{H}$ supposed to be Bockner integrable. $x(t)$ is the mild solution for restriction on a class of controls u given by

$$x(t) = x(t, u(.), x_0) = S(t)x_0 + \int_0^t S(t-\tau)A_0^{-1}Cu(\tau)d\tau. \tag{8}$$

The famous relation between exact controllability and complete stabilizability was first established by Slemrod who proved that the controllability from any state to any state implies the exponential stabilizability. In [6] Zabczyk showed that the implication in the opposite way is possible, some authors pricised that in the case of Hilbert spaces with bounded operators the previous idea is not avaible.

The system (7) is exactly cotrollable if for all x_0, $x_1 \in \mathcal{H}$, there exists a time T and a control $u \in L_p(0, T; U)$, $p \geq 1$, such that $x(T, u(.), x_0) = x_1$. For $x_1 = 0$ so, we talk about exact null controllability.

System (7) is said exponentially stabilizable, if there exists a linear feedback control $u(t) = Dx(t)$, where D another linear bounded operator such that

$$\|e^{A_0^{-1}(B+CD)t}\| \leq M_\omega e^{-\omega t}, \quad M_\omega \geq 1, \omega > 0. \tag{9}$$

The system (7) is said to be completely stabilizable if it is exponentially stabilizable for all $\omega > 0$.

Let $\mathcal{R}_T$ be the reachability operator defined as

$$\mathcal{R}_T u(.) = \int_0^T S(t-\tau)A_0^{-1}Cu(\tau)d\tau, \tag{10}$$

which is a linear bounded operator acting from L_p to $\mathcal{H}$. The system (7) is exactly controllable if $Im(\mathcal{R}_T) = \mathcal{H}$ also, it is exactly null controllable if and only if

$$Im(\mathcal{R}_T) \subset Im(e^{A_0^{-1}(B+CD)T}), \quad \forall T > 0.$$

We denote by $\mathcal{R}_T^*$ the adjoint operator of $\mathcal{R}_T$ satisfies the property:

$$\exists \delta > 0, \forall x^* \in \mathcal{H}, \|\mathcal{R}_T^* x^*\| \geq \delta \|x^*\|. \tag{11}$$

We can have many implicit conditions on exact controllability summarized by the bellow formulation:

$$\begin{aligned} \|\mathcal{R}_T^* x^*\| &= \left(\int_0^T \|C^*(A_0^*)^{-1}S^*(t)x^*\|^q dt\right)^{\frac{1}{q}}, \quad u \in L_p, p > 1, \\ \|\mathcal{R}_T^* x^*\| &= ess \quad sup\{\|C^*(A_0^*)^{-1}S^*(t)x^*\|\}, \quad 0 \le t \le T, u \in L_1; \end{aligned} \tag{12}$$

where $ess \quad sup$ is the essential supremum of a set.

Remark 2. *The condition of exact controllability in the class of controls L_p, $p > 1$ are equivalent but the situation become more complicated to show it in the class L_1.*

Theorem 5 ([7,8])**.** *The system (7) is exactly controllable in the class L_p if and only if*

- *For $p = 1$, if $\lim_{n\to\infty}(C^*(A_0^*)^{-1}S^*(t)x_n^*) = 0$, uniformly for all $t \in [0, T]$ then $\lim_{n\to\infty} x_n^* = 0$.*
- *$p > 1$, if $\lim_{n\to\infty}(C^*(A_0^*)^{-1}S^*(t)x_n^*) = 0$, for all $t \in [0, T]$ then $\lim_{n\to\infty} x_n^* = 0$.*

4. Conclusions

The notion of stability and controllability of newclass of controlled systems defined by implicit stationary differential equation was the purpose of this paper. In course of this work we used the spectral theory of the operator pencil $\lambda A - B$ to provide conditions for the stability in sens of Lyapunov also we studied the controllability of those systems. Furthermore, most of researchs will be directed to open problems for some particular types of systems on special spaces with infinite dimension.

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References

1. Doletski, Y.L.; Krein, M.G. *Stability of Solutions of Differential Equations in Banach Space*; American Math Society: Providence, RI, USA, 1974.
2. Favini, A.; Yagi, A. *Degenerate Differential Equations in Banach Spaces*; Marcel Dekker Inc.: New York City, NY, USA, 1999.
3. Rutkas, A.G. Spectral methods for studying degenerate differential–operator equations. *J. Math. Sci.* **2010**, 4246–4263.
4. Vlasenko, L.A. Evolutionary models with implicit and degenerate differential equations. *Dniepropetrovsk* **2006**.
5. Gantmacher, F.R . *Theory of Matrices*; Dunod: 1965; Volume 1,2.
6. Zabczyk, J. Mathematical Control theory: An introduction. *Syst. Control Found. Appl.* **1992**.
7. Olbrot, A.W.; Pandofli, L. Null controllability of a class of functional differential systems. *Int. J. Control* **1988**, 193–208.
8. Rabah, R. Controllability of linear systems with constant delay in Banach spaces. *RAIRO Automat.-Prod. Inform. Ind.* **1986**, 529–539.
9. Benabdallah, M. On the exponential stability of implicit systems in Banach spaces. *Chiers Math. de l'université d'Oran. Fascicule1* **1989**, 2–12.